# Identity *in* Physics: A Historical, Philosophical, *and* Formal Analysis

Steven French & Décio Krause

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# IDENTITY IN PHYSICS

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*A Historical, Philosophical, and Formal Analysis*

by

STEVEN FRENCH
and
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Brian to assembled crowd: 'We are all individuals.'
Crowd: 'We are all individuals.'
Brian: 'We are all different.'
Crowd: 'Yes, we are all different.'
A lone voice in the crowd: 'I'm not.'
*Monty Python's Life of Brian*

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# Preface

In a lecture given at Chicago University shortly after the development of quantum mechanics, Werner Heisenberg wrote the following:Many of the abstractions that are characteristic of modern theoretical physics are to be found discussed in the philosophy of the past centuries. At that time, these abstractions could be disregarded as mere mental exercises by those scientists whose only concern was with reality, but today we are compelled by the refinements of experimental art to consider them seriously.[1]

One such philosophical abstraction is the notion of 'individuality' and the impact of the new quantum physics on our understanding of the identity and individuality of objects was noted almost immediately by physicists such as Bohr, Born and Heisenberg himself. Our aim in this volume is to explore the nature and extent of that impact along historical, philosophical and formal dimensions.

Beginning with the historical, it was realized soon after the publication of Planck's famous 'black body' radiation paper that he had employed a form of statistical 'counting' quite different from that used by Boltzmann in the development of classical statistical mechanics. This new 'quantum statistics' became incorporated into subsequent developments through the work of Bose, Einstein, Fermi, Dirac and Heisenberg. The difference from Boltzmann's counting was understood as follows: whereas in classical statistical mechanics a permutation of two otherwise indistinguishable particles was taken to give rise to a new arrangement—because although indistinguishable, the particles were still distinct individuals—in quantum statistical mechanics it was not. This was taken to imply that the particles had 'lost' their identity and were, in some sense, non-individuals. How this implication was then accommodated metaphysically by the physicists and philosophers of the time is an interesting historical issue: Bohr, for example, incorporated it within the framework of complementarity, whereas the likes of Born and Eddington, on the physics side, and Cassirer, on the philosophical, took it as further support for a broadly structuralist ontology under which objects come to be 'reduced' to sets of invariants.

[1] Heisenberg 1930

However, as is now well known, this non-individuality of quantum particles is not a necessary consequence of quantum statistics. It turns out that one can accommodate the relevant physics and still regard the particles as individuals, but at a price. Something has to be given up, and what this is varies according to the metaphysical context adopted. Thus, for example, it has been argued that the particles, although individuals, are indiscernible in a way that violates Leibniz's famous Principle of the Identity of Indiscernibles. If this Principle is dropped then the individuality of the particles must be articulated in terms of something 'over and above' their relevant sets of properties, such as some form of substantial substratum perhaps.

If one were to regard these consequences as so unpalatable that the view of particles as non-individuals is to be preferred, then certain formal issues must be confronted. In particular, it is not clear how collections of non-individual objects can be captured by standard set theory. As the mathematician Yuri Manin put it: "[w]e should consider possibilities of developing a totally new language …" to deal with collections of entities which do not behave as standard sets (in the sense of obeying the axioms of the usual set theories), since the "[n]ew quantum physics has shown us models of entities with quite different behaviour. Even 'sets' of photons in a looking-glass box, or of electrons in a nickel piece, are much less Cantorian than the 'set' of grains of sand".

It is our intention in this book to explore these different issues and, in particular, to go some way towards developing the 'totally new language' suggested by Manin. We begin, in Chapter 1, by setting out some of the fundamental philosophical notions concerned with identity and individuality. In particular, we note a conceptual distinction, going back to the Scholastics, at least, between distinguishability and individuality. The distinction is important because even if two, or more, objects are understood to be indistinguishable, in terms of sharing all the relevant properties, say, one might yet want to insist they are distinct individuals, where this individuality is conceived in terms which 'transcend' the aforementioned properties. Alternatively, one might insist that individuality can only be understood in terms of some sets of properties and hence might be inclined to deny that two, or more, distinct objects can be truly indistinguishable. This last assertion amounts to a form of the Principle of Identity of Indiscernibles touched on above and we shall examine its status in some detail in subsequent chapters. Furthermore, this metaphysical framework has formal implications, as identity in higher-order logic is defined in terms of indistinguishability, and 'countable' objects which feature in set theory are typically taken to be those which have well-defined

identity conditions. Thus the framework introduced here underpins all our discussions in subsequent chapters.
In Chapter 2 we begin our exploration of modern physics with a historical survey of the classical treatment of collections of objects. In particular we examine the way in which an understanding of individuality underpins classical statistical mechanics and we spell out the way in which alternative metaphysical conceptions can be employed to obtain the required results. It is our contention that the nature and role of these conceptions has not been made clear or adequately appreciated in the recent literature. We also outline the impact of these kinds of issues on the foundations of space-time physics, again within the classical (non-quantum) context.

In the third chapter we turn our attention to the history of quantum statistics and our intention here is to indicate how a concern with the nature and implications of the 'new' statistics runs throughout this history. Thus Ehrenfest noted, early on, how Planck's new counting procedure, introduced at the heart of the 'old' quantum theory, differed from the classical form developed by Boltzmann. The philosophical implications of this procedure became apparent after the work that led to the two 'standard' forms of quantum statistics, namely Bose-Einstein and Fermi-Dirac (although as we indicate in our history, other forms are also theoretically possible). Principally this was taken to be that quantum particles should no longer be regarded as individuals, in some sense, a claim that became so ubiquitous that we have referred to it as the 'Received View'. How this implication was then incorporated into an appropriate metaphysical framework is then an interesting historical question in itself and we touch on the efforts of Bohr, Born and Schrödinger in this regard.

However, in the next chapter we argue that the Received View is not a necessary consequence of the physics and that one can, in fact, maintain that quantum particles are individuals, subject to certain constraints. We explore the sense of 'individuality' that might be appropriate here and in particular we argue that any such sense based on Leibniz's Principle of Identity of Indiscernibles is undermined by the physics. Nevertheless, this leaves other possibilities and we conclude that there exists a form of 'metaphysical underdetermination' between two conceptual packages which are both supported by the physics: particles-as-individuals and particles-as-non-individuals. This raises interesting issues for realism in the quantum context.

Even if quantum particles are regarded as individuals, they still present metaphysical problems of course. In particular, they appear to be individuals

to which labels can be attached but for which these labels are essentially otiose. In Chapter 5 we consider a range of philosophical positions on the nature and role of names in physics. This takes us into a discussion of the work of Dalla Chiara and Toraldo di Francia, who have presented a non-standard form of set theory known as 'quaset theory', which they claim is capable of accommodating the implications of quantum statistics for particle individuality.

Of course, Dalla Chiara and Toraldo di Francia's formal approach is not the only one available. In Chapter 6 we consider what we call 'The Manin Problem', namely, the search for an 'adequate' language for talking of 'sets' of elementary particles. In discussing this issue, we outline the way in which classical logic and mathematics (set theory) encompass the concept of identity. Finally, we also consider how indiscernibility is dealt with within such classical frameworks, as preparatory background for incorporating the Received View within a formal account.

Dropping the very notion of individuality presents problems for such a formal account which even Dalla Chiara and Toraldo di Francia's quaset theory cannot deal with. Hence in Chapter 7 we present quasi-set theory as 'the mathematics of non-individuality'. The principal ideas of this theory were developed by one of us (Krause), independently of Dalla Chiara's and Toraldo di Francia's work. Various formal results are presented and it is shown how quantum statistics can be accommodated within this set-theoretical framework. Of course, this represents only the first step towards Manin's 'adequate language', but we hope the reader will both appreciate its strengths and gain a clear indication as to how we might take this approach further.

The logical counterpart of this approach is then presented in Chapter 8. Here we present a kind of logical system in which the Principle of Identity is not valid in general. As the name suggests, this is based on the ideas of Schrödinger, as discussed previously, and here we consider the first-order system originally presented by da Costa, before moving on to higher-order systems. Since identity can be defined in such systems, this provides a formal framework for discussions of the Identity of Indiscernibles. One of the systems presented encompasses modalities, being a kind of 'intensional logic'. We also introduce a new kind of predicate, 'quantum sortal predicates', and their relationships with sortal predication and sortal logics are discussed.

Finally, in Chapter 9 we extend our discussion to include the metaphysics of quantum field theory, outlining the recent debates over the status of particles within that theory. In particular we note that Teller's metaphysics of 'quanta'

might be fruitfully accommodated within our framework and we suggest how the Fock space formalism can be developed within quasi-set theory.

The basis of this work has previously been published in a variety of papers and we are grateful to the following publishers for permission to use this material: The American Institute of Physics, Analysis, Cambridge University Press, Editions Hermes-Lavoisier, The Journal of Philosophy, Kluwer Academic Publishers, National Centre for Logical Investigation, Notre Dame Journal of Formal Logic, Oxford University Press, Princeton University Press, Sociedade Paranaense de Matematica, and the Stanford Encyclopedia of Philosophy.

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Steven French

Décio Krause

# 1
# Introduction

> I wonder if I've been changed in the night? Let me think: was I the same when I got up this morning? I almost think I can remember feeling a little different. But if I'm not the same, the next question is 'Who in the world am I?' Ah, that's the great puzzle!
> Lewis Carroll, *Alice in Wonderland*

Most of us would agree that the chairs, trees, rocks, umbrellas, people and many of the so-called 'everyday' objects we encounter can be regarded as individuals. How is this individuality to be understood? A first attempt at answering this question might point out that these objects are all distinguishable from one another. This distinguishability can then be understood—apparently quite straightforwardly—in terms of differences in the properties of the objects and it seems reasonable to suggest that these differences also provide the basis for the objects' individuality. But what if it is possible for two or more such objects—two umbrellas, say—to possess all the same properties? Is there any way of ensuring this cannot happen? We could, for example, build into our philosophical account of individuality some fundamental principle which states that no two objects can possess the same set of properties. This is effectively what Leibniz did, of course, with his infamous 'Principle of the Identity of Indiscernibles': two objects which are indistinguishable, in the sense of possessing all properties in common, cannot, in fact, be two objects at all. In effect the Principle provides a guarantee that individual objects will always be distinguishable.

And of course most, if not all, of our everyday objects will possess some distinguishing property, some nick or scratch that will distinguish them even from similar objects of the same shape, colour, etc. But again, it seems possible,

at least, for two such objects—our two umbrellas again—to possess not only the same shape and colour, but even the exact same nicks and scratches. What then? In answer we might point to some property which cannot be shared, such as location in space-time. Clearly—or so it would seem—our two umbrellas cannot occupy the same space at the same time. But why not? Because—it might be answered—they are impenetrable. Pushing this line even further, one might seek some ground for this impenetrability in the stuff, the substance, of which the objects are, in some sense, 'made'. This meshes with quite a different answer to our initial question: the individuality of the objects is to be understood not in terms of some property or set of properties which also render them distinguishable, but in something else, underlying the properties, such as some form of substantial substratum, famously characterized by Locke as the 'something we know not what' (since it cannot be described in terms of the properties it is supposed to underpin).

There is, of course, a huge philosophical literature on the various attempts to answer these and related questions and provide what is often referred to as the 'principle' of individuality. Our concern in this work is in a certain sense more straightforward: can the accounts sketched above be extended to the fundamental objects posited by current physical theories, such as electrons, protons, neutrons etc.? Can these be regarded as individuals, like rocks and people? Or are they more like drops of water or money in a bank account? How do we begin to answer these questions? Obviously we must turn to the relevant physics, namely quantum theory, and, in particular, we must consider what this physics tells us about the aggregate behaviour of these objects. Do they behave, in aggregate, like rocks or people, or like water drops? This behaviour is described by the relevant statistical mechanics and so we shall need to consider the implications of this quite carefully.

One response to our concern, which was first articulated very shortly after the development of quantum statistical mechanics, is that quantum particles cannot be regarded as on a par with rocks and people, that they are, in some sense, 'non-individuals'. More recently, the alternative view has been defended that, on the contrary, quantum particles can, in fact, be regarded as individuals, albeit with very different properties and behaviour from their classical counterparts. This gives rise to a very interesting situation in which our fundamental metaphysics appears to be underdetermined by the physics. Our primary aim in this book is to explore the nature and consequences of these metaphysical views and, by doing so, put some flesh on the bare bones that have been laid down in the literature.

We shall begin with an outline of various important distinctions concerning distinguishability, individuality and identity. As we have already said, there is an enormous literature on these topics which we cannot hope to do justice to here but we hope to set out a clear framework in which our further discussions can take place. We shall then review the history and fundamentals of both classical and quantum statistical mechanics, before outlining the arguments for each of the above metaphysical views. Each of these will be considered in turn. The view of particles as non-individuals has traditionally been articulated in terms of a lack of self-identity, typically explicated in terms of suggestive metaphorical comparisons with money in a bank account or waves on a rope. We believe we can go further than such metaphors and shall offer formal accounts of 'quantum non-individuality' in terms of 'Schrödinger logics' and 'quasi-set theory'. This view has also been claimed to mesh with Quantum Field Theory (QFT) and we shall argue that the 'quanta' of QFT can likewise be approached by quasi-set theory.

Although the alternative view, of particles as individuals, can draw on the traditional metaphysical accounts of individuality, the non-classical behaviour of quantum particles still has important philosophical implications within this context. Thus we shall consider the status of the Principle of Identity of Indiscernibles, reviewing well known arguments as to its necessity or contingency. The conclusion we shall come to is that, given a certain understanding of the relevant physics, it is contingently false. This claim has recently been disputed and we shall look carefully at some of the arguments that have been put forward in this debate. Since it is the infamous quantum 'entanglement' which lies at the heart of these discussions we shall also consider the nature of this entanglement from the perspective of this view of the particles as individuals. Our suggestion will be that it can be understood—metaphysically—in terms of so-called 'non-supervenient' relations holding between the particles and we shall explore the nature of the latter. This will put some 'flesh' on the bones of this particular metaphysical package, so that we have two reasonably well worked out alternatives: abandon individuality for quantum particles and understand the notion of 'non-individuals' within the formal framework of quasi-set theory; or retain a sense of individuality in the absence of the Identity of Indiscernibles and accept a non-standard metaphysics of relations.

By the end of the book we hope to have shown, at the very least, that there are fundamental issues here—formal, physical and metaphysical—and to have indicated that there is still a great deal of interesting and important work to be done.

# 1.1 INDIVIDUALITY, DISTINGUISHABILITY, AND IDENTITY

Let us begin by outlining some philosophical distinctions. In a work such as this, there is always the question of exactly how much philosophical detail to go into. Too little, and we run the danger of blithely skating over important issues; too much, and we may find ourselves hopelessly lost in a thicket of metaphysical minutiae. Hopefully, we have found the right balance here.

We shall begin with a notion that will figure prominently in our discussions, namely that of identity. It has been said that this is 'utterly simple and unproblematic'[1] since everything is identical to itself and nothing is identical to anything else. However, this blunt assertion can be understood as a reaction to the convoluted philosophical wrangling over this notion following the work of Frege, in particular, and continuing to the present day. Consider a statement of the form '*a* is identical to *b*', written symbolically, '$a = b$', such as, to take a famous historical example, 'Hesperus is identical to Phosphorus'. Such statements apparently assert that a relation holds, but between what?

According to one view (associated with the Frege of the *Begriffsschrift*), the identity relation holds between *names*, so two names, 'Hesperus', 'Phosphorus', stand in that relation if and only if they refer to the same object. The sentence 'Hesperus is identical to Phosphorus' then expresses a (meta-linguistic) thought about linguistic entities, that is, the names. As is well known, this view appears to be supported by reflection on the difference between the statements '$a = a$' and '$a = b$'. The former appears to hold *a priori*, whereas the latter may represent an advance in our knowledge (as when we conclude that 'Hesperus is identical to Phosphorus'). But then how can there be such an advance when the relation concerned holds between linguistic entities, rather than the objects themselves?

Such concerns motivate the alternative view (associated with the Frege of 'On Sense and Reference'[2]) that the identity relation holds between *objects*, so two (putatively) objects stand in that relation if and only if they are the same object. The above sentence then expresses the thought that Hesperus and Phosphorus are the same object. On this view, the names 'Hesperus', 'Phosphorus' express different senses but refer to the same object and hence

[1] Lewis 1986, pp. 192–3.

[2] However, this apparent shift in Frege's position has been disputed; see Caplan and Thau 2001.

we can understand how identity statements of the form 'Hesperus is identical to Phosphorus' can be informative.

There is, of course, much more that we could say here and some of it we will say in later chapters but for the moment we will follow the latter Frege and insist that to state that 'item *a* is identical to item *b*', written symbolically, 'a = b', means that there are not two distinct items in reality, but only one, which may be referred to as either *a* or *b*. In particular, looking ahead to our discussions of the situation in physics, we shall take the statement 'Particles *a* and *b* are identical' to mean that there are not two particles at all but only one, referred to by different names. As we shall see, many commentators—physicists and philosophers alike—understand such statements as meaning that the particles are what we shall call indistinguishable. It is important to establish the terminology we shall use at this early stage. As far as we are concerned, the difference in names of *a* and *b* reflects at most an epistemological difference,[3] rather than an ontological one.

Now what about the statement '$a = a$'? This asserts that *a* is identical to itself and hence it can be understood as an expression of the *self-identity* of the item concerned. It is this which is typically taken to be simple and unproblematic.[4] Why it is unproblematic depends on which of the above views one adopts. If one takes identity to hold between names, then as we indicated above, '$a = a$' is true *a priori*. If, on the other hand, identity is understood as holding between objects, then '$a = a$' expresses a fundamental relation an object has with itself, one which surely cannot be denied.[5] In either case, '$a = a$' appears to be a logical truth and with '=' formalized as a two-place predicate in the language of classical first-order logic, '$a = a$' is typically taken to be a fundamental postulate of classical logic with identity (often referred to as the 'Principle of the Reflexivity of Identity'). However, this is in fact more problematic than it seems. In particular, as we shall argue, if self-identity is tied to the notion of individuality and quantum entities are understood as objects which are non-individuals, then such entities fail to be self-identical and an alternative formal framework is required to accommodate them.

Of course, a number of fundamental issues immediately arise, of which we shall be particularly concerned with two: first, the sense in which self-identity

[3] Of course, the different names need not even reflect that; they may simply be indicative of different modes of presentation of the object concerned.

[4] Thus Lewis insists that the standard problems concerning, for example, the sense (!) in which an *F* is identical to a *G* can all be restated without identity (op. cit., p. 193).

[5] As Lewis put it, "There is never any problem about what makes something identical to itself; nothing can ever fail to be." (ibid., pp. 192–3).

can be understood as a relation and second, the manner in which it is represented formally. As we shall see, the two are not unrelated. Before addressing these issues, we need a little more conceptual apparatus.

First of all, 'Item' is used above as a neutral word that encompasses both universals and particulars. We shall refer to particulars that exist in the physical world as physical individuals or just individuals for short. Some particulars that one would naturally regard as individuals persist through time; others do not. Examples of the former include the particles of physics that we will primarily be concerned with; examples of the latter include space-time points, which we shall briefly consider, and instantaneous temporal parts under certain philosophical interpretations, which we shall not discuss. Those individuals that persist through time are referred to as continuants and can be said to possess *trans-temporal identity*. In these terms, physical individuals can be distinguished from other sorts of particular such as events or states of affairs that may be said to occur or obtain, rather than to persist (in the case of particles) or exist (in the case of space-time points). Individuals which have more or less well-defined spatial locations are typically referred to as 'things'.

Secondly, we shall insist on a conceptual distinction between *individuality* and *distinguishability*. Some may question the basis for such a distinction but, as we shall demonstrate in this work, a number of issues can be clarified by making it. We can approach the distinction by focusing on the particularity, or non-instantiability, of individuals, as contrasted with the universality, or instantiability, of their attributes.[6] The former pertains to something, some 'principle of individuality', as it is often called, which is 'internal' or 'particular' to the individual in the sense of being associated with it alone.[7] The latter involves the individual's relationship with other particulars and is therefore 'external' in some sense. One way of articulating the distinction is to imagine a possible world in which there is only one entity; this entity cannot be regarded as distinguishable from others—or so it is claimed—since there are no other entities, yet it may surely be considered to be an individual.[8]

[6] See, for example, Gracia 1988, Ch. 1.

[7] Thus, for example, the Scholastic philosopher Suarez referred to the 'fundamental unity' of the individual.

[8] Gracia ibid. This example is not unproblematic, of course. Hacking has insisted that specifications of possible worlds—our 'imaginings'—must make reference to a spatio-temporal background as part of such a world (Hacking 1975). If this is accepted then one might press the point that distinguishability, here understood in terms of the relationship between the entity and the spatio-temporal background 'external' to it, and individuality cannot be conceptually cloven. Of course, this demands a particular view of possible worlds and of the relationship between logical and physical possibility, one which Hacking himself has articulated but which is debatable nevertheless.

Thus we shall take—unhelpfully perhaps—'individuality' to be that in virtue of which an individual is an individual and 'distinguishability' to be that in virtue of which an individual is distinguishable from other individuals. We say 'distinguishable' rather than 'distinct', since the numerical *distinctness* or *diversity* of entities can be understood in terms of either their distinguishability or their individuality.[9]

Three fundamental questions now arise:

(1) What confers individuality on physical individuals?
(2) What confers distinguishability on such individuals?
(3) What confers trans-temporal identity on these individuals?

It might be argued that (1) is actually an empty question since there is simply no need for any conferring 'Principle' if individuality is taken as primitive. On such a view, it is not conferred by or grounded in anything else, whether—to jump ahead a bit—some subset of properties or something metaphysically beyond such properties. And on such a view, distinctness or diversity would also be primitive; that still leaves the question as to what renders an individual distinguishable from others. However, taking a concept as primitive runs the danger of losing the unificatory coherence and understanding offered by conceptual reduction. At the very least it needs to be argued that such reduction is so problematic or unhelpful that we should turn our backs on it. More importantly, as far as we are concerned, taking individuality as primitive will fail to illuminate certain fundamental issues that we are interested in and, as we shall see in Chapter 2, will lead to confusion. At this stage, we ask for the reader's indulgence: let us kick off with this conceptual framework of individuality and distinguishability and see how far we can take it. Returning to our questions, it may be that the answer to (1) can also serve as an answer to (2) and (3), in the sense that that which confers individuality upon an entity turns out to be the same as that which renders it distinguishable from other entities but what we want to insist on, at this point, are the conceptual distinctions involved.

To say what an entity's individuality *consists in*, in terms of this framework, is first to say that it consists in something having to do—metaphysically speaking—with that entity and no other and secondly to say what precisely that something is. To tackle the latter, we need to address question (1). Possible

[9] As we shall see, the possibility arises in physics of entities which are indistinguishable yet still distinct individuals.

answers can be broadly divided into two kinds: (a) those that appeal to some subset or 'bundle' of the properties of the entity, and (b) those that appeal to something else, 'over and above' these properties. Let us consider each in turn.

## 1.2 'BUNDLE' INDIVIDUALITY

'Principles' of individuality that involve sets or 'bundles' of properties or attributes must face the problem of multiple instantiability which can be expressed in the further question: what is it that guarantees that some other entity could not possess the same set or subset of properties? Failure to come up with such a guarantee would be disastrous for this approach. One response is to invoke some set or subset of properties, together with some further principle which ensures that no other entity could possess that set or subset. Thus, one might take the relevant properties to be or include spatio-temporal ones and invoke the Impenetrability Assumption (IA), to the effect that no two entities can occupy the same spatial location at the same time; that is, the points of space-time are either monogamous or virginal.[10] As we shall see, this is a common response to this issue not only in philosophy but also in (classical) physics. More generally, the requisite guarantee has been sought in Leibniz's Principle of Identity of Indiscernibles (PII), which insists that there cannot be two, or more, individuals which have all relevant properties in common.

This principle occupies a pivotal position in the debate between the adherents of positions (a) and (b) above. Not surprisingly there has been considerable discussion regarding its status. On the one hand, it has been regarded as necessary, in the sense that the very supposition that there might be two things exactly alike in all their properties is self-contradictory. Leibniz himself suggested that it followed from his 'Complete Notion of an Individual',[11] according to which the notion of an individual contains all the predicates that individual will possess. By definition, no two individuals can possess the same Complete Notion or, equivalently, the same set of properties. Such a view motivates the 'bundle' theory by effectively resolving individuality into the (complete) set of properties.[12]

[10] Quinton 1973, p. 17.

[11] See Russell 1964, p. 433; this notion is referred to by Broad as the 'Predicate-in-Notion principle'; Broad 1975, p. 6.

[12] Of course, Leibniz himself would have regarded this as entirely too crude, committed as he was both to the notion of substance, regarded as that which can only be a subject, not a predicate, and the metaphysics of the *Monadology*, according to which the ultimate individuals are not physical but rather immaterial monads.

Elsewhere,[13] he argues that it follows from his Principle of Sufficient Reason, since if there were two indiscernible objects, they would have to be created in different places but God would have no reason for choosing between the two alternatives that arise through considering a permutation of the particles. This suggests that the Principle is regarded as contingent and Leibniz does present the image of someone scurrying about the garden unable to find two leaves exactly alike. Of course, in response to the claim that, nevertheless, we can imagine two such indiscernibles, Leibniz retorts that such a situation does not constitute a 'genuine' possibility, suggesting that the necessary/contingent distinction is perhaps too crude for this form of metaphysics.

Leaving the historical exegesis aside, there are several well-known arguments to the effect that the Principle cannot be necessary, based on similar sorts of imaginings.[14] Under the pressure of these arguments, supporters of the Principle have staged a retreat to the less forceful claim that it is only contingently true[15] and that, even if no longer regarded as an *a priori* logical truth of metaphysics, it may still be a useful methodological principle.[16] It is precisely the former claim that we will be examining in the context of classical and quantum physics in later chapters.

Now, indiscernibility, or, equivalently, indistinguishability,[17] is here understood in terms of the mutual possession (however that is understood) by the individual (however *that* is understood!) of the relevant properties. The question as to which are the relevant properties obviously assumes a crucial importance and different forms of the Principle can be conceptually delineated depending on the set chosen. Rewriting it in terms of second order logic with equality, it can be expressed thus:

$$\forall F(F(a) \leftrightarrow F(b)) \rightarrow a = b \tag{1.2.1}$$

[13] In the correspondence with Clarke; see Alexander 1956, pp. 36–45 and p. 62.

[14] The most famous of these invites us to consider a possible world containing two indiscernible iron globes (Black 1952). Hacking spurns the invitation on the grounds—indicated above—that the description of this particular kind of possible world is incomplete since it leaves out the spatio-temporal background (Hacking 1975). By appealing to the non-Euclidean geometry of the latter, Hacking argues that the Leibnizian can avoid the force of Black's counterexamples—by redescribing the possible world in terms of another equivalent one involving one globe and Riemannian space-time, for example. However, this attempt founders on a misunderstanding of the relevant physics, in terms of which the equivalence cannot in fact be maintained (French 1995).

[15] See, for example, Casullo 1984.

[16] See, for example, Hoy 1984.

[17] It has been suggested that indiscernibility should be regarded as an 'in principle' notion, whereas indistinguishability concerns what we can discern at a given time (Hilborn and Yuca 2002). Since we are more concerned with metaphysical issues here we shan't appeal to this further distinction.

where *a* and *b* are any two individual terms and *F* is a variable ranging over the possible attributes of these individuals.

The question is, what sort of attributes should be included in the range of *F*?

If the attribute 'being identical with *a*', which is certainly true of *a*, is included, then PII becomes a theorem of second order logic.[18] Leaving self-identity aside for the moment, different forms of PII, of increasing logical strength, can be delineated, according to the kinds of attributes *F* is taken to range over. Thus the weakest form, which we shall label PII(1), states that it is not possible for there to exist two individuals possessing *all* their properties (relational and non-relational) in common, whereas what we shall call PII(2) excludes those which can be described as spatio-temporal from this set. An even stronger form, PII(3), results if the set is taken to include only monadic, non-relational properties.[19]

It has been argued that of the three, PII(1) must be necessary, since no two individuals can possess exactly the same spatio-temporal properties or enter into exactly the same spatio-temporal relations.[20] However, this obviously involves the Impenetrability Argument (IA) as an implicit premise and, as we shall see, the status of the latter in the context of modern physics is contentious.[21] Furthermore, both PII(1) and (2) allow relations into the bundle and therefore carry the implication that these may confer individuality. Many philosophers vigorously dispute this on the grounds that, to cite one form of argument, since relations presuppose numerical diversity, they cannot account for it.[22] If such arguments are accepted then PIIs (1) and (2) must both be abandoned and the strongest form alone admitted, with the variable *F* covering strictly monadic properties only. Of course, this exclusion of relations from the bundle only applies to those that are irreducibly relational. It is commonly agreed that Leibniz himself believed that all relations—particularly spatial

[18] A formal account will be given in Chapter 6.

[19] This hierarchy can be compared with that given in Quinton op. cit., pp. 24 –5 or Hoy op. cit., p. 276. It has been argued that some sort of distinction along these lines is required if the bundle programme is to be defended from its critics and that it then becomes entirely more plausible if based on what we have called PII(1), with relations granted the capacity to individuate; see Casullo op. cit. and the discussions in Armstrong 1978, p. 92–5 and in the essays collected in Loux 1976.

[20] Quinton op. cit., p. 25.

[21] Of course, if I A is found to be violated, then a defender of PII might want to argue that the entities involved should not be regarded as individuals in the first place. We shall discuss this possibility of protecting PII from violation later.

[22] See for example Russell 1956 and Armstrong op. cit., pp. 94–5. Hoy suggests that this smacks of unwarranted prejudice against relations but fails to deal with the argument itself (op. cit., p. 277). As we shall see, this possibility of allowing relations to individuate has been resurrected in the quantum context.

ones—could be reduced to monadic properties, thereby collapsing all forms of the Principle into PII(3).[23] Again, it is not our intention to delve into the history here; we mention this issue simply because of the possibility that physics presents us with examples of such irreducible relations.

The standings of PIIs (2) and (3) do not depend on IA and so it is possible to drop either or both without having to abandon the latter assumption. Examples of individuals which are indistinguishable in the sense of possessing a common set of non-spatio-temporal properties would obviously be enough to show that PII(2) is contingently false. If this set includes monadic properties then PII(3) will obviously be false as well. Equally obviously, however, if the individuals concerned possess only relational but non-spatio-temporal properties then PII(3) is saved. As we shall see, this touches on the further issue of how we regard properties in physics.

As we have said, the 'bundle' view of individuality needs some form of PII to guarantee individuation. Shifting to the claim that the Principle is contingently true leaves this view in a precarious situation and invites consideration of the physical nature of the world in which it is supposed to hold. We shall examine the impact of these considerations in what follows.

## 1.3 TRANSCENDENTAL INDIVIDUALITY

Principles of individuality which invoke something over and above some set of properties of an entity are examples of what Post has called 'Transcendental Individuality' (TI).[24] As originally introduced, this notion is somewhat ambiguous. The example given was that of losing one's umbrella and being faced in the lost property office by a number of umbrellas which, crucially, are indistinguishable in the sense already mentioned of possessing their non-spatio-temporal properties in common. Yet, it was insisted, the question 'but which is *my* umbrella?' still makes sense. Individuality, it was claimed, must lie with something over and above the set of properties in terms of which the umbrellas could be regarded as indistinguishable.

[23] This interpretation has been disputed (Hintikka 1976), although Leibniz did emphasize, outside of this correspondence with Clarke, the importance of non-spatio-temporal properties in distinguishing two apparently indiscernible things (Leibniz 1896, 27.1 and 27.3).

[24] Post 1963. The terminology is unfortunate in having Kantian overtones. The siren sound of such overtones should be resolutely ignored.

Put like this, one candidate for this TI would be the subset of spatio-temporal properties already mentioned. Indeed, we could determine which was our umbrella, at least in principle, by reconstructing the spatio-temporal trajectory of the thing and following it back to us. Obviously such a determination would fail if IA were violated. And again, as we have already said, this view can be found in philosophical reflections on classical physics. We shall call it 'Space-Time Individuality', or STI for short. Of course, such a view immediately invites speculation as to the nature of the individuating spatio-temporal background. On a relational view of space and time, the spatio-temporal locations of physical individuals involve relations with other individuals and a possible circularity develops.[25] This may be broken by invoking a privileged set of continuants by reference to which all other things are individuated. But the nature of this privilege must be spelled out and the danger is that STI collapses into some other account of individuality, such as the one sketched below. Alternatively, the circularity could be broken by adopting a substantivalist view of space and time according to which spatio-temporal properties and relations of physical things are reduced to the properties of space-time points. Of course, this in turn requires some explication of what it is that confers individuality upon the points of space-time themselves.[26]

As an alternative to STI, we could take Post's notion to refer to something 'transcending' *all* the properties of a thing, including the spatio-temporal ones. Historically, various candidates have been proposed for this 'something', including 'haecceity', 'primitive thisness', 'fundamental unity' and 'substance'. Having separated off STI, we shall reserve the abbreviation T I to describe this whole collection of candidates. We accept that this obscures the fundamental differences between them but what is important for our purposes here is the central idea that individuality is conferred by *something* that goes beyond the properties; the metaphysical nature of that something is of less concern, although it will be touched upon.

This answer to our question (1) must face the problem of describability: if description, in its positive form, involves a listing of attributes, how can we describe that which 'transcends' these attributes? All we can do, it seems, is describe this something in negative terms. Thus, as we have already noted, Locke famously and problematically described his substantial substratum as

[25] Of course, if we focus on (instantaneous) spatial locations only, then the two umbrellas can still be individuated in terms of their temporal relations to some object.

[26] We shall return to this issue in Chapter 2.

a "something, we know not what". Empiricist minded philosophers have not taken kindly to such notions and have preferred an answer in terms of (a) above. The status of PII then becomes crucial.[27]

Formally we get around (or, some might say, obscure) the describability problem by introducing proper names or labels or variables, *x*, *y*, etc. which stand for or designate the individual, with predicates representing properties, as we have in our characterization of PII above. As is well known, how we understand such names or labels is philosophically contentious: running roughshod over the subtleties, the 'Descriptivist' theory insists that proper names are only shorthand for definite descriptions, a view which meshes nicely with response (a) above; the 'Non-Descriptivist' theory argues that proper names simply refer to objects and that naming is not merely a disguised form of describing, a view that might be seen as going hand in hand with TI; and the 'causal' view attempts to marry elements of these by agreeing that names refer to objects but that this reference is initially established via some form of 'baptism' (either descriptive or ostensive) from which a causal chain extends to uses of the name. Again, we shall not enter into the pros and cons of these views here but we will return to these distinctions later.

Having briefly outlined some of the metaphysical issues and touched upon the introduction of names, or labels, we can now consider the question of how we might express the notion of 'non-individuality' in these (formal) terms. The problem is obvious: if this 'transcendental' something or other is literally indescribable, then how are we going to express the lack of it?! A way out of the dilemma is to understand the above notion of 'primitive thisness' in terms of 'self-identity'. Thus Adams takes primitive thisness as "… the property of being identical with a certain individual—not the property that we all share, of being identical with some individual or other, but my property of being identical with me, your property of being identical with you, etc.".[28] Extending this insight, the idea is apparently simple: regarded in haecceistic terms, 'Transcendental Individuality' can be understood as the identity of an object with itself; that is, '$a = a$'. We shall then defend the claim that the notion of non-individuality can be captured in the quantum context by formal

[27] One could always sidestep the problem by insisting that individuality is primitive. If it is not conferred by or grounded in anything, then it would not be surprising that we have difficulty articulating what this ground might be.

[28] op. cit., p. 6. Cf. Heidegger 1957, p. 26. Auyang, whose concerns match ours, writes: "Identity does not say anything beyond one thing; rather, it discloses the meaning of being an entity, and the disclosure signifies our primordial understanding" (Auyang 1995, p. 125).

systems in which self-identity is not always well defined, so that the reflexive law of identity, namely, $\forall x\,(x = x)$, is not valid in general.

## 1.4 SELF-IDENTITY

Let us briefly consider this notion in a little more detail. If self-identity is understood as a property or, more specifically, a relation, like any other then it would seem that we have returned to a form of answer (a) above, albeit in a rather extreme form. In particular, the Principle of Identity of Indiscernibles then becomes true, if vacuous. However, such an understanding is problematic. Barcan Marcus, for example, suggested that "Individuals must be there before they enter into relations, even relations of self-identity".[29] However, talk of 'entering into' is surely misplaced in this case: it seems apt in the case of an individual 'entering into' a relation, such as parenthood for example, but not in the case of an individual entering into the relation of identity with itself. In the former case the relation can be formed only when there is more than one individual; the establishment of the relation is conceptually posterior to the existence of the relevant individuals. In the latter case, however, the existence of the individual and the establishment of self-identity are conceptually on a par in that we cannot envisage the possibility of one without the other. An individual is thus conceptually tied to its identity with itself in a manner in which it is not with other relations (and it is for this reason, perhaps, that identity is dismissed as unproblematic).[30]

This intimate relationship between individuality and identity has also been related to a further notion we shall mention later, that of countability. Thus, Lowe characterizes an individual as:[31]

> (…) an object that is differentiated from others of its kind in such a fashion that it and they are apt to constitute a *countable plurality*, with each member of such a plurality counting for just *one*, a *unit* of its kind.[32]

[29] Barcan Marcus 1993, p. 20.

[30] 'Instantiation', which is an external connection between a property and what instances it, presupposes that a full-fledged entity fit to be the subject is already constituted. If such a subject cannot be an individual aside from instantiating the property of being an individual, this would make the account of individuation in terms of this property circular and useless (op. cit., p. 137).

[31] This understanding can also be found in Russell's approach, for example: ' "How shall we define the diversity which makes us count objects as two in a census?" We may put the same problem in words that look different, *e.g.*, "What is meant by 'a particular'?" or "What sort of objects can have proper names?" ' (Russell 1948, p. 292).

[32] Lowe 1994b, p. 536. Lowe has recently modified this by substituting '*determinately distinct*' for 'differentiated' (private correspondence).

Furthermore, it is a necessary condition of countability that

> … the items to be counted should possess determinate identity conditions, since each should be counted just once and this presupposes that it be determinately distinct from any other item that is to be included in the count.[33]

Thus, on this view, if a plurality is countable, the entities of which it is composed must possess self-identity. However, 'countable' is ambiguous, between cardinality and ordinality. It is only the latter which requires determinate distinctness.

We shall return to this further distinction later, as it will prove crucial for our discussion of quasi-set theory. For the present we simply want to emphasize that conceiving of individuality in terms of self-identity will allow us to appropriately represent its denial.

This brings us on to our representation of identity. In logic and mathematics the concept of identity is introduced in different and non-equivalent ways, depending on the language employed. In first-order languages, it is common to take a binary predicate as primitive, satisfying the reflexive law and the substitutivity principle.[34] The standard semantics then interprets that predicate in the diagonal of the considered domain; in other words, it is intended that the chosen predicate represents identity in the domain, although such a diagonal cannot be completely characterized by first-order languages. In higher-order languages, identity can be defined by the so-called Leibniz' Law.[35] Intuitively, this definition says that 'two' objects are identical (that is, are the very same object) if and only if they share all the same properties, that is, if and only if they are indistinguishable.[36]

## 1.5 DISTINGUISHABILITY

At last we come to the issue of distinguishability. As it is not susceptible to the problem of describability noted above, distinguishability is obviously easier

[33] Lowe op. cit., p. 536.

[34] Mendelson 1997. In some first-order languages, as in set theory, the relation of identity can be defined; see Chapter 6.

[35] Cf. the definition of identity in Whitehead and Russell 1925, as we shall see.

[36] Wittgenstein rejected this way of introducing identity, precisely because it did not allow for the possibility of indistinguishable objects, a possibility, moreover, which appeared to be allowed by the form of logical atomism on which *Principia Mathematica* was based (we are grateful to Roger White for pointing this out). As we have said, these more formal considerations will be mentioned in Chapter 6. Ramsey also questioned Whitehead and Russell's definition, saying it is not contradictory at all for two things to have all the same properties (Ramsey 1965, p. 30). As we shall see, the definition of identity due to Hilbert and Bernays is not vulnerable to these objections.

to characterize. We recall question (2) above: what is it that confers distinguishability on individuals? Answers have typically involved either (i) one or more of the set of properties of the entity or (ii), spatio-temporal location. If PII is secure, then option (i) will guarantee individuality in the sense of position (a) above. If impenetrability holds, then (ii) likewise guarantees individuality in the sense of STI. Our conceptual distinction between individuality and distinguishability can then only be maintained in practice under view (b) above; that is, by adopting some form of T I. In this case, it can be said that either spatio-temporal location or a certain set of properties allow one to *infer* individuality, but that what *confers* it, metaphysically speaking, is something like the Lockean substratum or primitive thisness, underlying, or 'transcending', the properties of the individual. It is by means of the distinguishability or determinate distinctness of an object that we become aware of it as an individual, in a negative fashion as Gracia puts it. However, ontologically and positively, as an individual it is one thing, or, as Suarez said, it has an 'individual unity'. In terms of these distinctions we can understand the process of naming or labelling an individual as involving two aspects: the first concerns our epistemic access by means of which we become *aware* of the individual, the second is to do with the act of denoting, or giving the individual a name. The former involves description, whereas the latter may not.[37] On this view, the primary function of (proper) names is to refer and they are established through an act of baptism. On the other hand, we *learn to use* these names effectively via description. Thus, a description may play an epistemic role in distinguishing an individual from others, in terms of some set of properties, and in fixing reference through baptism, but may not be necessarily tied to the name.[38]

With this distinction between distinguishability and individuality in hand, we can understand the different senses in which individuals can be indistinguishable. As we have indicated, two, or more, individuals are regarded as indistinguishable if they possess the same (sub)set of properties. The extent of this subset will give us different degrees of indistinguishability and also, as we have already set out, different forms of PII. Here we can usefully introduce yet another distinction, that between intrinsic and extrinsic properties. Again, this is hotly debated in the philosophical literature but for our purposes we can articulate it by introducing the notion of the state of a thing. In the vernacular, we might comment on the 'state' of our umbrella, referring to its battered

[37] Gracia op. cit., p. 219.

[38] ibid., p. 220.

and torn appearance for example. Physical systems can be in certain states, typically characterized by such properties as velocity, energy etc. Such properties can be called 'state-dependent' and may be spatio-temporal, although not necessarily. We shall find it useful to identify the intrinsic/extrinsic distinction with that between state-independent and state-dependent properties.[39]

On the STI view, then, two individuals can be indistinguishable in terms of their intrinsic properties but not in terms of their spatio-temporal ones, if impenetrability is assumed. Whether physics allows for such individuals and whether PII is violated, and in what form, is something we shall consider in subsequent chapters. If we accept some form of TI, then there would appear to be no metaphysical impediment to having individuals which are 'absolutely' indistinguishable in the sense of possessing both their intrinsic and extrinsic properties in common. This may appear peculiar, since we could not distinguish such individuals, not even by their different states, yet it is permitted in terms of our conceptual distinction above. Again, whether physics allows for individuals in *this* sense is something to be discussed below.

The metaphysical peculiarity of TI is further revealed if we consider that such individuals may be in *possible* states and, indeed, it is in these terms that the impact of physics has been cast. Bringing the arcane machinery of possible worlds to bear, TI would appear to support the doctrine of haecceitism.

## 1.6 HAECCEITISM

As originally introduced, this is the doctrine that holds "… that it does make sense to ask—without reference to common attributes and behavior—whether *this* is the same individual in another possible world, that individuals can be extended in logical space (*i.e.* through possible worlds) in much the same way we commonly regard them as being extended in physical space and time, and that a common 'thisness' may underlie extreme dissimilarity or distinct thisness may underlie great resemblance …".[40] Lewis has considered whether

[39] This distinction may further be used to explicate the notion of 'natural kinds': objects with the same set of intrinsic properties form such a kind. This notion, like so many here, has been widely discussed in the philosophical literature. We shall simply note that the above distinction meshes well with scientific practice in the form of the classifications used by physicists themselves. Such classifications provide a good measure of support for the idea of 'theory-dependent natural kind terms' as defended by Fales 1979; for further discussion see French 1985.

[40] Kaplan 1975, p. 722. Lewis expresses it thus: "If two worlds differ in what they represent *de re* concerning some individual, but do not differ qualitatively in any way, I shall call that a *haecceitistic difference*. *Haecceitism*, as I propose to use the word, is the doctrine that there are at least some cases of haecceitistic difference between worlds. *Anti-haecceitism* is the doctrine that there are none." (Lewis op. cit., p. 220). The scholastic notion of *de re* representation raises the issue of how this representation is to be effected. An alternative approach to haecceitism is to understand it in terms of the identity conditions for possible worlds: there can be non-qualitative differences between worlds and such worlds are not identical. Such differences could then be articulated via the various forms of 'Transcendental Individuality', as we shall indicate below. We are grateful to James Ladyman for suggesting (something like) this.

a belief in haecceities is either necessary or sufficient to maintain haecceitism and has concluded that it is neither. However, this conclusion hinges on a particular construal of haecceity and his own philosophical view of properties or attributes. Thus he insists that one might expound haecceitism, without believing in haecceities, where the latter are understood as non-qualitative properties (such as self-identity)—one might be a nominalist, for example, and reject properties altogether.[41] Of course, such a nominalist then owes us an account of individuality sufficient for maintaining haecceitism. On the other hand, he argues, one might reject the doctrine but accept haecceities, again in the sense of a non-qualitative property (where properties in general are identified with sets of possible individuals). The further acceptance of haecceitism is then avoided by giving an account of trans-world identity in terms of his theory of counterparts.[42]

Clearly there are important issues here, to do with how we understand the very notion of a property and also that of a possible world. We shall merely note that T I meshes well with haecceitism but that one could accept the latter and not haecceities in the sense of primitive thisness, or the non-qualitative property of self-identity, if one were to understand T I*à la* Locke, for example, in terms of substance.[43]

Could one accept haecceitism and understand individuality in terms of ST I? Presumably in order to talk of the same individual in different possible worlds on such a view one would have to impose a constraint on these possible worlds such that the underlying space-time background is the same. Obvious problems arise with regard to the suggestion that one might talk of the same individual in, say, worlds with Euclidean and non-Euclidean space-times.[44]

[41] Lewis op. cit., p. 225.

[42] Ibid.

[43] Can the same substance exist in different possible worlds and possess different properties in those worlds? A positive answer would suggest a view of possible worlds which overlap at the level of the substantial substratum. Lewis has argued against overlap views on the grounds that they involve the contradictory assignment of properties to individuals (Lewis op. cit., pp. 198–210). Such arguments appear to have no force against this version of the overlap view. Thus one could maintain haecceitism without haecceities in *this* sense and without having to be a nominalist.

[44] We recall again Hacking's insistence that any specification of a possible world must involve a specification of this spatio-temporal background.

Furthermore, the Impenetrability Assumption would have to be given a modal spin to ensure that it holds appropriately across possible worlds. In other words, one would have to consider the transworld identity of space-time points. Leaving this esoterica aside, we have set out the bare bones of haecceitism here because some have found it useful in considering the impact of physics on the metaphysics of individuality. We shall briefly return to it in Chapter 2.

Finally, let us now turn to our third question: what is it that confers trans-temporal identity upon individuals?

## 1.7 TRANS-TEMPORAL IDENTITY

This concerns the problem of re-identification through time; that is, of supplying the grounds on which an individual *b* at time $t_2$ can be identified as the same individual *a* at an earlier time $t_1$. A necessary condition of trans-temporal identity would seem to be spatio-temporal continuity.[45] In other words, we can say that *a* is identical with *b* only if there is a continuous trajectory connecting them,[46] or, in other words, that *b* and *a* should both be 'individual stages' in the succession of such stages which corresponds to the 'career' of a single persisting individual. This trajectory or career must then satisfy the following necessary conditions:[47]

I. The trajectory or succession must be spatio-temporally continuous.[48] However, there are an infinite number of continuous spatio-temporal paths between any two space-time points separated by a non-vanishing interval. Thus it has been suggested that a criterion of qualitative continuity is required.

II. The trajectory or succession must be qualitatively continuous in the sense that any individual stage on the trajectory or in the succession must be qualitatively similar to a neighbouring individual stage on the trajectory or in the succession.[49] This is still not enough, as I and II together would not prevent us from tracing an individual in such a

[45] Shoemaker 1963.

[46] Quinton op. cit., p. 66.

[47] Possibly the best discussion of these conditions is given in Hirsch 1982, esp. p. 36.

[48] A detailed analysis of this criterion in terms of the set-theoretic concept of the continuum is given by Coburn 1971.

[49] See, e.g., Quinton op. cit., and Hirsch op. cit., pp. 10–15.

way that we combine stages of the individual with stages of some of its parts.[50] What is further required is a sortal constraint such that every individual stage in the succession falls under a sortal term *S*.[51]

III. There is a sortal term *S* such that the succession is a succession of *S*-stages or the trajectory underlies such a succession.[52]

Of course, the *S*-stages themselves must be momentary in the sense that they do not depend on any criteria for reidentification through time.

Other conditions have also been suggested such as the requirement that the matter of which a thing is composed at one moment should be the same, or nearly the same, as the matter of which it is composed at the next moment and that there must be a causal relationship between individual stages.[53] It is not our intention to become involved in a detailed discussion of these further requirements here but it is worth noting that the causal criterion may obviously be related to I above.[54]

In this context we can once again see the difference between the approaches to individuality delineated above. STI is more economical, metaphysically speaking, in the sense that it claims that it is the points of space-time (however they are understood) which *confer* individuality, distinguishability and trans-temporal identification. However, this conferral hinges on the Impenetrability Assumption[55] and any threat to the latter would undermine the whole edifice. According to the TI view, on the other hand, the points of space-time have only a secondary role to play in allowing us to *infer* a trans-temporal identification and it is something else, the underlying Lockean substratum or the primitive thisness of the object, which, ontologically speaking, *confers* the re-identifiability. And, as we shall see, the way in which we formally represent the notion of non-individuality will have an impact on our understanding of sortal terms and how they themselves are formally represented.

[50] Hirsch op. cit., pp. 32–3.

[51] As Hirsch indicates, it may be difficult to draw a clear distinction between what does and does not count as a sortal term; examples include car, tree, rock and other nouns. For a detailed discussion of sortal terms see Wiggins 1967; see also Chapter 8.

[52] Underlying this constraint is the less exact but conceptually more basic idea that for a succession of thing stages to correspond to stages in the career of a single persisting thing, the succession must minimize change—except of course spatio-temporal change! See Hirsch op. cit., pp. 72–82.

[53] See, e.g., Shoemaker 1979.

[54] See, e.g., Lucas 1984.

[55] At the very least it is difficult to argue for the possibility of trans-temporal re-identification and *against* I A.

## 1.8 SUMMARY

At the heart of our discussion lies a conceptual distinction between individuality and distinguishability: the former relates to the individual taken on its own, whereas the latter is concerned with the individual's relationship with others. So-called 'bundle' theories of individuality effect an ontological economy by accounting for both individuality and distinguishability in terms of some set of properties. Given the multiple instantiability of distinguishing properties, such theories then need to appeal to the Principle of Identity of Indiscernibles in order to guarantee individuation. However, the status of this Principle turns out to be problematic in the context of modern physics.

An alternative approach also accounts for distinguishability in property terms but then insists that individuality must be understood through something else, 'over and above' or transcending the properties. The nature of this something else varies, from the familiar Lockean substratum to some form of haecceity or 'primitive thisness'. The last has been further understood in terms of a particular and rather special property of things, namely their self-identity. The denial of this notion, both metaphysically and formally, will then allow us to get a grip on the concept of 'non-individual' particles.

Both approaches to individuality and distinguishability allow us to take individuals as *nameable*, although the conceptual analysis of these names will of course vary, *countable*, where this is tied in with self-identity, and *re-identifiable*.[56]

Having presented this plethora of distinctions, we shall now explore the metaphysical foundations of both classical and quantum physics in terms of this overall framework.

[56] With regard to Transcendental Individuality, Redhead and Teller have extended the work of French and Redhead 1988 and made the further distinction between 'Label Transcendental Individuality' (LT I) and 'Property Transcendental Individuality' (1991; 1992). It is in virtue of the former that individuals are nameable, countable and re-identifiable and it is in virtue of the latter that they bear properties. Teller (1995) prefers to understand LT I in terms of 'primitive thisness' and in his later work (1998) he invokes a minimalist sense of haecceity as that which involves self-identity. Given this latter emphasis, it is not surprising that many of Teller's concerns match ours, although we disagree, as we shall see, on the issue of whether only one form of metaphysics can be unambiguously read off from the physics.

# 2
# Individuality in Classical Physics

> I see no reason why energy shouldn't also be regarded as divided atomically.
> Boltzmann, Halle Conference 1891

Let us begin by recalling the distinction between *distinguishability*, understood as involving more than one object and *individuality*, understood as pertaining to that object taken by itself. Epistemologically, it is through distinguishability that we become aware of something as an individual. Ontologically we may then go on to analyse that individuality either in the same terms or via some other conceptual principle, such as Suarez's 'individual unity' or some form of Transcendental Individuality, for example. Now this is all very well for 'macroscopic' objects, to which we have relatively straightforward epistemic access, but what about their microscopic counterparts, such as atoms and electrons? These are not observable, or at least, not in the way that tables, rocks and other people are, and if we cannot get a similar epistemic grip to begin with, how are we to proceed to consider their individuality?

The answer, as Reichenbach correctly realized, is to look at how such objects behave collectively and, from that epistemic standpoint, come to some conclusion regarding their individuality. In other words, we replace the examination of the object itself with inferences based on the *statistical* behaviour of an assemblage of such objects.[57] Thus, consider the distribution of two particles of the same kind—two electrons for example—over two 'boxes' or energy states, where the particles are *indistinguishable* in the sense we noted in the previous chapter; that is, in the sense of possessing the same intrinsic properties such as (rest) mass, charge, etc. We can generate the following arrangements.

In classical physics, (3) (see Fig. 2.1) is given a weight of twice that of (1) or (2), corresponding to the two ways the former can be achieved by permuting

[57] Reichenbach 1956, pp. 228–9.

Figure 2.1. Two particles over two energy states.

the particles. This gives us four combinations or complexions in total and hence we can conclude that the probability of finding one particle in each state, for example, is 1/2. This probability assignment effectively characterizes Maxwell-Boltzmann statistics, the historical origins of which we will briefly consider below. As we shall see in the next chapter, we obtain quite different probability assignments when it comes to quantum statistics. Note that it is assumed that none of the four combinations is regarded as privileged in any way, so each is just as likely to occur.[58]

The question now arises: on what grounds do we assign (3) a weight twice that of (1)? Such grounds cannot involve the *distinguishability* of the particles, since they are indistinguishable, in the above sense. Hence, it has been concluded, the increased weight given to (3)—that is, the counting of particle permutations—is accounted for by the claim that the particles are individuals. In other words, from this aspect of their collective behaviour, it can be inferred that the particles must be individuals, since their permutation makes a difference to the counting. Thus, it is typically concluded, classical particles must be regarded as individuals, in some sense.

Now, this conclusion can be resisted. One might insist that the 'extra' weight given to (3) is just some kind of 'brute fact', for which no account should be given. However, that response effectively blocks any further metaphysical exploration of the foundations of classical statistical mechanics. A more effective response would be to point out that there may be grounds, other than the statistics of the particles, for not counting their permutations, thus removing any need to account for the weight given to (3). The infamous Gibbs Paradox, for example, suggests that on thermodynamical grounds such permutations need to be 'factored out' of the relevant mathematical expressions.[59] We shall defer a full consideration of this response for two reasons: first

[58] The justification of this assumption is a well-known problem in the foundations of classical statistical mechanics and we shall touch on it again in what follows.

[59] If the permutations are not counted, a concern arises as to how—on this view—we can ensure that we get the right (that is, Maxwell-Boltzmann) statistics. A possible way of meeting this concern is given in Huggett 1999, and we shall discuss this work later in the chapter.

of all, we need to delineate the foundations of classical statistical mechanics a little further and set down these mathematical expressions before we tackle the paradox itself. Secondly, the effectiveness of this response in general hinges on one's understanding of the import of the paradox: does it reveal something profound about the foundations of classical statistical mechanics or was it a harbinger of the shift to an alternative form of statistics, underpinned by quantum mechanics? Those who are keen to resist the claim that classical particles are individuals can be understood as answering 'yes' to the first question, while we prefer to take the second option. Hence we will leave this response for now and return to it at the end of the chapter, before moving on to quantum physics.

If the argument from permutations itself is accepted, there is the further question: how should this individuality be understood? As we shall see, different answers to this question can be given in terms of the views given in Chapter 1. However, we shall argue, the physics itself does not unequivocally support one of these answers over the others, contrary to what is often suggested in the literature.

This form of underdetermination of the metaphysics of individuality was reflected in the historical development of classical statistical mechanics, so we shall digress to consider the context in which the balls and boxes argument is put forward. This will not only allow us to present the relevant physical framework in terms of which recent discussions of these issues have been presented, but it will also reveal how the metaphysics featured in the construction of the theory itself. What Reichenbach's manoeuvre effectively does is to give us epistemic access to the particles by taking us down the famous reduction of the macroscopic properties of thermodynamics to the microscopic ones covered by statistical mechanics. Let us now consider this route in more detail.

## 2.1 A BRIEF HISTORY OF CLASSICAL STATISTICAL MECHANICS

The kinetic theory of matter attempts to explain the empirical regularities occurring in the macroscopic properties of material objects in terms of the microscopic behaviour of their atomic and molecular constituents as described by Newton's laws of mechanics. The early studies in the revival of this theory which occurred in the second half of the nineteenth century had only limited

explanatory power since they assumed that (1) these constituents all had the same velocity; and (2) the influence of intermolecular forces and impacts were negligible.[60] The development of the subject took a major leap forward when Maxwell allowed intermolecular collisions to play a role and demonstrated that their effect was to produce a statistical distribution of molecular velocities in which all velocities would occur with a known probability.[61] This is given by Maxwell's famous distribution law:

$$f\left(x\right) = \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2} \qquad (2.1.1)$$

where $a$ is a constant and $x$ is the component of the velocity in the $x$-direction.

Of course, the statistical element was introduced purely as a matter of convenience in order to overcome the practical problems in handling enormously large numbers of atoms and molecules. The latter themselves were regarded as traversing distinct, continuous space-time trajectories and as obeying Newton's laws.

The original proof of the above law was widely regarded as unsatisfactory and subsequent attempts to derive it more rigorously can be divided into three types:

(I) If a Maxwellian distribution is already established then it can be shown that conservation of energy implies that further collisions will leave it unchanged. Hence this distribution is the only one which is stable. This was the approach taken by Maxwell himself.[62]

(II) One may define a quantity, $H$, which depends on the velocity distribution and then show that the effect of molecular collisions is always to decrease $H$, unless the distribution is Maxwellian, in which case $H$ remains constant. This constitutes the essence of Boltzmann's '$H$-Theorem' approach.

(III) One may regard the atomic and molecular velocities as random quantities and obtain the best possible estimate of their distribution, subject to the constraints of fixed energy and number of atoms or molecules, using probability theory. Maxwell's derivation may be improved by calculating all the possible ways of dividing the energy among the atoms or molecules, subject to the above constraints. This

[60] See Brush 1976.

[61] Maxwell 1860.

[62] Ibid., Vol. II, p. 26.

was the thinking which subsequently developed into the 'Combinatorial Approach' (and which lies behind the permutation argument above).[63]

Approaches II and III rest on very different foundations: the *H*-Theorem Approach is explicitly based on a consideration of molecular trajectories, whereas the Combinatorial Approach eschewed such detailed consideration and was simply concerned with the distribution of molecules over energy states. Historically, the Combinatorial Approach played an absolutely fundamental role in the development of quantum theory.

Maxwell's work had an enormous impact upon Boltzmann, who was seeking a mechanical explanation of the apparent irreversibility of natural processes as expressed by the Second Law of Thermodynamics. Put rather crudely, this states that it is impossible for heat to flow from a colder to a hotter body. The law was expressed by Clausius in 1865 in terms of his newly introduced entropy function *S*, where

$$dQ \leq T \cdot dS, \tag{2.1.2}$$

where $dQ$ is the heat change, $T$ is the temperature change and $dS$ is the change in entropy.[64] The equality holds for reversible processes, of course. The task facing Boltzmann was then two-fold: first, to demonstrate the existence of an entropy function satisfying Clausius's definition and secondly to show that this function could only increase in an irreversible process. After reading Maxwell's 1867 paper, Boltzmann realized that the key lay with the above distribution function and in 1871 he was able to demonstrate the existence of an entropy function in purely mechanical terms and lay down a procedure for finding it.[65] The following year he took the next step and analysed the behaviour of his entropy function in irreversible processes by considering the way in which the velocity distribution changed with time due to intermolecular collisions.[66] This work can be understood as both an attempt to complete the statistical mechanical reduction of the Second Law, and as showing that the effect of intermolecular collisions on a gas in a non-equilibrium state would be to drive it to equilibrium as described by Maxwell's law. Boltzmann was able to show that the Maxwellian distribution represents the equilibrium state and obtained an explicit formula for the rate of change of $f$, $\frac{\partial f}{\partial t}$, on the basis of

[63] One may also treat the Maxwellian distribution as a primitive postulate of the theory and as justified on the basis of its explanatory and predictive power, although we shall not consider this here.

[64] Clausius 1865.

[65] See Daub 1969.

[66] Boltzmann 1868, Vol. I, p. 316.

an "… exact consideration of the collision process" between two molecules of a spatially homogenous,[67] low-density gas with no external forces present. With this formula in hand, he was able to show that *f always* tends towards the Maxwellian form. To do this he introduced an auxiliary quantity $E$ (which later came to be called $H$),[68] defined by

$$E = \int_0^\infty f\left(xt\right)\{\log[f\,(x,t]\,/\,\sqrt{x}\,] - 1\}\,dx$$

(2.1.3)

By considering the symmetrical character of the collision and the possibility of inverse collisions, and assuming, as Maxwell had before him, that the velocities of the two molecules before they collide are statistically independent,[69] Boltzmann demonstrated that $E$ could only decrease with time;[70] that is,

$$\frac{dE}{dt} \leq 0$$

(2.1.4)

With $H$ substituted for $E$, this corresponds to Boltzmann's $H$-Theorem. $E$ cannot decrease to infinity, of course, and so the distribution function must approach a form for which $E$ has a minimum value and its derivative vanishes. At this value the function has, and can only have, the Maxwellian form.

Thus −E increases in the irreversible approach to equilibrium and hence behaves like the entropy function. Furthermore, −E was actually proportional to the entropy in the equilibrium state. This implies, as Boltzmann himself indicated, an entirely new approach to proving the Second Law, one that could deal with the increase in entropy in irreversible processes as well as with its existence as an equilibrium state function.[71] In this way, the $H$-Theorem effectively extended the definition of entropy to non-equilibrium situations and completed the kinetic reduction of the Second Law of Thermodynamics.[72]

Now, although Maxwell's distribution function introduced a certain statistical 'coarse-graining' into the description, the development of the $H$-Theorem was still based upon consideration of binary collisions between molecules

[67] Ibid., Vol. I, p. 329.

[68] Brush suggests that it was introduced through a combination of educated guesswork and trial and error (Brush 1976, Vol. II, p. 351 and p. 449).

[69] This is the problematic randomness assumption known as the '*Stosszahlansatz*'.

[70] The proof is quite straightforward and relies on the fact that the quantity $(a-b)\log b/a$ is always negative if $a$ and $b$ are real positive numbers; Boltzmann op. cit., pp. 335–45.

[71] op. cit., pp. 345–6.

[72] See Brush op. cit., p. 443; Klein 1970, p. 102.

traversing distinct continuous trajectories. Thus, despite the probabilistic elements, the underlying basis was still, of course, Newtonian and deterministic. The tensions this caused within Boltzmann's edifice are well known.[73]

In particular, his conclusion, that the E/H-function always decreased, was disputed by Maxwell, Tait and Thompson[74] and, independently and famously, by Loschmidt, who noted that in any system, "… the entire course of events will be retraced if at some instant the velocities of all its parts are reversed".[75] If entropy is a specifiable function of the positions and velocities of the particles of a system, and if that function increases during some particular motion of the system, then reversing the direction of time in the equations of motion will specify a trajectory through which the entropy must decrease. For every possible motion that leads towards equilibrium, there is another, equally possible, that leads away and is therefore incompatible with the Second Law. Loschmidt concluded that if the kinetic theory were true, then the Second Law could not hold universally and thus Boltzmann's proof of the *H*-Theorem could not be correct.

Boltzmann's 1877 response to the Loschmidt 'paradox' is illuminating. He basically admitted that one could not, in fact, prove that entropy increased 'with absolute necessity' and that, according to probability theory, even the most improbable non-uniform distribution is still not absolutely impossible.[76] However, he then claimed that the existence of such improbable entropy-decreasing situations did not contradict the fact that for the overwhelming majority of initial states the entropy could be counted on to increase and that the improbabilities associated with the former case were, for all practical purposes, impossibilities. Boltzmann now focused on the probabilistic aspect

[73] $f$ is coarse-grained in the sense that a given form is compatible with an infinite number of different possible arrangements of the particles. Now, in carrying through his 1872 derivation, Boltzmann wrote $f$ in a form explicitly dependent upon time, developed a differential equation for its time dependence and treated its form at $t = 0$ as an initial condition which the appropriate solution for the equation must satisfy. However, this treatment is inappropriate for a statistical function of the system's coordinates as the coarse-grained nature of $f$ actually prevents it from functioning as an initial condition in the strict classical sense. Each of the different possible arrangements of particles which it allows correspond to a different initial condition for a completely specified mechanical system and each results in a different trajectory for that system over time. The actual form given to $f$ at $t = 0$ does not therefore determine its form in later times (except for those times so close to $t = 0$ that few, if any, molecular collisions take place). The equations for $f$ and $E/H$ actually determine only average or most probable values and many, many other rates of change are compatible with the given initial distribution of the particles of the system.

[74] See Klein 1970, p. 84 and Daub 1970, p. 213.

[75] Loschmidt 1876, p. 139.

[76] Boltzmann 1968, Vol. II, p. 112. Similar conclusions were reached by Maxwell, Tait, Thompson and Gibbs, the latter writing that "… the impossibility of an uncompensated decrease of entropy seems to be reduced to an improbability" (Gibbs 1961, Vol. I, p. 167).

of the *H*-Theorem and argued that it followed from this that the number of states leading to a uniform, equilibrium distribution after a certain time must be much larger than the number leading to a non-uniform distribution, since, he claimed, there are infinitely many more uniform states than non-uniform ones.[77] No justification was actually given for this last claim, although Boltzmann intimated where one might be found:

> One could even calculate the possibilities of the various states from the ratios of the number of ways in which their distributions could be achieved, which could perhaps lead to an interesting method of calculating thermal equilibrium.[78]

This 'interesting method' was subsequently elaborated later that same year in one of the most significant papers of classical statistical mechanics, entitled 'Probabilistic Foundations of Heat Theory'.

Prior to this Boltzmann had, in a series of papers from 1868 and 1871, re-derived and extended Maxwell's results and, significantly, in 1868, sketched an alternative derivation that was free from any assumptions regarding inter-molecular collisions.[79] Thus he considered the distribution of a fixed amount of energy over a finite number of molecules in such a way that all combinations were equally probable. By regarding the energy as divided into small but finite packets, Boltzmann could treat this as a problem in combinatorial analysis. In this manner he obtained a complicated expression which reduced to Maxwell's law in the limit of an infinite number of molecules and infinitesimal energy elements. This marks the beginnings of his Combinatorial Approach.

In the 1877 work, Boltzmann drew on this earlier work and presented a new and radical alternative to the *H*-Theorem approach.[80] In line with his general philosophical attitude towards physical theories[81] this 'Combinatorial Approach' was elaborated through a succession of models of increasing complexity and closer approximation to the physical situation.

The first such model was a highly simplified and explicitly fictional discrete energy model in which he considered a collection of molecules whose individual energies were restricted to a finite, discrete set, with the total energy held fixed. If $\omega_k$ is the number of such molecules with energy $k\varepsilon$, then the

[77] Ibid., p. 122.

[78] Ibid., p. 120.

[79] Boltzmann 1868, Vol. I, p. 49.

[80] Boltzmann 1968, Vol. II, p. 164. We do not see this work as merely a continuation of the *H*-Theorem approach as Klein suggests (Klein 1970, p. 77).

[81] See Boltzmann 1905. A useful discussion of Boltzmann's philosophy of science can be found in Cercignani 1998, pp. 184–97.

set $\omega_0, \omega_1, \ldots, \omega_p$ is sufficient to define a particular macro-state (*Zustandverteilung*) of the gas.[82] Boltzmann then noted that such a macro-state could be achieved in many different ways, each of which he called a 'complexion'.[83] In general, if a complexion was specified by a set of numbers, each fixing the energy $k_{i\varepsilon \text{ of the } i}$th molecule, then, he wrote, a second complexion belonging to the same macro-state would be achieved by any permutation of the two molecules *i* and *j* which have different energies. Thus a permutation of particles between different energy states gave rise to a different macro-state of the system as a whole.

The number of such complexions for a given distribution can be found using well-known combinatorial techniques:

$$\mathcal{P} = \frac{n!}{\omega_0!\ \omega_1! \ldots} \tag{2.1.5}$$

where $\mathfrak{p}$ is the 'permutability' of the macro-state. The most probable such macro-state is then found by maximizing $\mathfrak{p}$ subject to tzhe constraints on total energy and number of particles.

As an illustration, Boltzmann invoked the now classic image of a large urn filled with numbered slips, the number on each slip standing for the number of energy elements to be assigned to a particle. A drawing of all the slips determines a complexion and the most probable state will be that for which $\mathfrak{p}$ above is a maximum. Thus what Boltzmann did was to take the 'permutability' measure as proportional to the probability of a distribution $\omega_0, \omega_1, \ldots, \omega_p\}$, or, strictly, the logarithm of the probability, as we shall see. To be exact he set the probability $\omega$ equal to the ratio of $\mathfrak{p}$ for a given distribution to the sum of all values for all allowed distributions.[84] It is important to note that when he took the number of complexions compatible with a given distribution as a measure of the probability of that distribution, Boltzmann emphasized that any particular complexion was as likely to occur as any other; that is, all complexions are equally probable.

Boltzmann continued his analysis by noting that finding the most probable state by maximizing $\mathfrak{p}$ was equivalent to minimizing the denominator in the expression above, since *n* is fixed. Minimizing the logarithm, for the sake of computational ease, and using standard techniques, it is straightforward to conclude that for $p \gg n$, $\mathfrak{p}$ will be a maximum if the occupation numbers

[82] See Boltzmann 1905, p. 188.

[83] Ibid., p. 169.

[84] Ibid., p. 169.

are given by

$$\omega_k = \frac{n\varepsilon}{\mu} e^{\frac{-s\varepsilon}{\mu}} \tag{2.1.6}$$

where μ is the average energy of a molecule and *s* is a constant (which came to be known as 'Boltzmann's constant'). This specifies the most probable distribution, which is what Boltzmann was after.

Of course, the energy elements (ε) were regarded by Boltzmann as nothing more than a convenient device and the next step was to move to a (classically) more realistic model with continuous energies. This was based on a division of the energy continuum into finite intervals $k\varepsilon$ *to* $k(\varepsilon + 1)$, with the occupation numbers given by $\varepsilon f(k\varepsilon)$, *where f* is the molecular distribution function. Proceeding much as before but going to the limit as $\varepsilon \to 0$ and with sums replaced by integrals, the most probable distribution is given by the familiar Maxwellian form.[85] However, this is for two, rather than three, dimensions and to obtain the Maxwellian distribution in the latter case, one must divide up the three-dimensional velocity space, rather than the energy continuum, thus giving Boltzmann's third model.[86] He was able to show that maximizing a form of ℘ in this context was equivalent to maximizing an expression Ω—the 'permutability measure'—which was just the negative of the *H*-function introduced five years previously. Since he had already shown then that *H* reaches a minimum when *f* corresponds to the Maxwellian distribution he felt that he did not need to repeat this here. This completed his demonstration that the case of thermal equilibrium corresponded to the most probable state of the gas.[87]

The next step was to extend the Combinatorial Approach to non-equilibrium situations and explain irreversible behaviour using the same principles. As Boltzmann emphasized, the permutability measure Ω is well defined whether or not the situation is in equilibrium, and hence could serve as a suitable generalization of the entropy.[88] In the equilibrium case, the behaviour of Ω must match that of the entropy as given by the Second Law; that is, it must either increase or, for reversible processes, remain constant. This characteristic was then extended to transitions between non-equilibrium states,

[85] Ibid., p. 190; see also Kuhn 1978, p. 50.

[86] Boltzmann op. cit., pp. 191–4.

[87] However, as Boltzmann himself realized, in going over from the fictional discrete to the more realistic continuous case, the previous basis for the assignment of equal probabilities, or statistical weights, to the complexions has been lost (ibid., p. 195). A new basis was introduced but this turned out to be problematic; we shall return to this shortly.

[88] Ibid., p. 218.

which do not conform to the Maxwellian distribution, and for which the entropy had not previously been well defined. Boltzmann pointed out that Ω *was* well defined for both classes of states and that it "… can always be computed; and its value surely will be necessarily greater after the change than before".[89]

Boltzmann then asserted as a general theorem, that for an arbitrary change between states that need not be characterized by equilibrium,

> … the total permutability measure of all the bodies will increase continuously during the change of state and it can at most remain constant if all the bodies throughout the transformation approximate thermal equilibrium infinitely closely (a reversible change of state.[90]

This was Boltzmann's reformulation of the Second Law as the statement that systems go from less to more probable states, thus extending the meaning of entropy into contexts where it was thermodynamically ill defined.[91] Einstein subsequently called this the 'Boltzmann Principle' and took it to express the idea that entropy increase could be interpreted as an increase in *disorder*.

Now, what was the point of this historical digression? It was to bring out two fundamental aspects of Boltzmann's work. The first is tied up with his *H*-Theorem Approach of 1972. As we have seen, although a crucial Maxwellian statistical element was introduced via the *f* function, this was ultimately grounded on a consideration of molecular trajectories. Given that the molecules are all indistinguishable in the sense of possessing the same intrinsic or state-independent properties, some further principle of individuality is required in order to even talk about distinct trajectories in the first

[89] Boltzmann, p. 209.

[90] Ibid., p. 218.

[91] See, e.g., Klein op. cit., p. 82. It has been pointed out—first by Planck and more recently by Kuhn (op. cit., p. 53)—that there is a circularity in Boltzmann's arguments here. After demonstrating that the state of thermal equilibrium is the most probable state of the gas and establishing the relationship between Ω and the equilibrium entropy, Boltzmann argued that for systems in equilibrium Ω must increase or remain constant because it was known from the Second Law that the entropy must do so. As an argument to establish the behaviour of a mechanical concept on the basis of thermodynamical considerations this is perfectly valid, but not if it is taken in reverse and used to justify the claim that a mechanical explanation of the Second Law has been provided. The above claim that the value of Ω will *necessarily* be greater after a change needs further support which Boltzmann fails to give. It may be that the relationship between the permutability measure and the probability of a distribution convinced him that his arguments were stronger than they actually were. Certain statements in this 1877 paper and the later 'Gas Theory' (Boltzmann 1910, pp. 38–47; it is significant that in this later work the step involving the permutability measure is omitted) suggest that he had in mind the plausible idea that all natural changes proceed from states of low probability to those of higher probability.

place. Boltzmann, more philosophically minded than many of his contemporaries, was completely aware of this issue. In his treatise on the principles of mechanics, the very first axiom of mechanics states that indistinguishable particles which cannot occupy the same point of space at the same time can be individuated by the initial conditions of their trajectories and the continuity of their motion. This, Boltzmann insisted, "… enables us to recognize the same material point at different times".[92]

Here we see the assumption of a form of 'Space-Time Individuality', as described in our Introduction, which then feeds into an understanding of the transtemporal identity of the molecules.[93] As well as the obvious emphasis on the continuity of the space-time trajectories, it is also important to note that 'impenetrability' is presupposed.

The second aspect is associated with the alternative 'Combinatorial Approach'. This eschews an explicit consideration of molecular trajectories in favour of that of the distribution of energy elements among molecules. Again the latter, although indistinguishable, are regarded as individuals; indeed, it might be said that from the classical perspective they could hardly be regarded otherwise, and, significantly, this is expressed in the theory via the counting—in the expression for $\wp$ and hence also in that for $\Omega$—of a *permutation* of these molecules. Through such a permutation we obtain a new 'complexion', although the macro-state of the gas remains the same. It is this form of counting which lies at the heart of Maxwell-Boltzmann statistics and which, with the Combinatorial Approach in general, was absolutely crucial for the development of quantum statistics and quantum theory in general, as we shall see. It has also provided the basis for much of the discussion of the individuality of particles in physics, via the argument given at the beginning of this chapter. With trajectories apparently left out of the picture, however, it might seem that the form of individuality associated with the counting of complexions obtained by permutations cannot be that of Space-Time Individuality. Indeed it has been argued that some other form of Transcendental Individuality (TI) must be involved here,[94] such as the Lockean kind, for example. However, as we shall see shortly, this argument can be undermined, since it can be demonstrated that such a form of TI is not *necessary* in this context.

[92] Boltzmann 1897–1904, Vol. I, p. 9; see also Boltzmann 1974, p. 227.

[93] See, for example, the brief discussion in Cercignani op. cit., p. 166, where it is noted that Boltzmann's axiom "… acquires a particularly pioneering aspect, because it shows what we understand by classical identical particles, as opposed to the indistinguishable particles of modern quantum mechanics" (ibid.).

[94] See, e.g., Post 1963.

Before we consider that, however, there is another important feature regarding the Combinatorial Approach which should be highlighted, since it sheds further light on the difference between this and the *H*-Theorem approach. This is to do with the justification for the claim that all complexions are equally probable. As Boltzmann himself realized, in going over from the discrete to the continuous case in his succession of models, the previous basis for the assignment of equal probabilities, or statistical weights, to the complexions, was lost. A new procedure is required and Boltzmann introduced the assumption that equal weights were to be assigned to cells of equal volume in the velocity space. This was underpinned by an appeal to a form of Liouville's theorem, which states that if the molecules are contained within such a cell at time $t = 0$, then with arbitrary forces acting on them, they will remain in the image of this cell which has the same volume as the gas evolves in time.[95] Thus no such cell in the velocity space is privileged.

However, this move is problematic since the molecules of the gas will not remain in the same volume of velocity space as time goes on but will in fact disperse throughout this space.[96] Here we need to introduce an important distinction between the 6-dimensional phase space whose coordinates are the components of molecular position and momentum (three each of course) and the 6N-dimensional phase space spanned by the position and momentum vectors of the *N* particle gas (the former was called 'μ-space' and the latter 'Γ-space', by the Ehrenfests).[97] In the former, the gas is represented by an assembly of *N* points and the distribution of these phase points over the space is given by Maxwell's distribution law. The distribution function *f* gives the density of particles in μ-space, in the sense that the number of molecules which are located in the volume element $d^3\hat{r}d^3\hat{p}$ constructed around the phase point $(r, p)$ at time *t*, is equal to $f(r, p, t)d^3\hat{r}d^3\hat{p}$. It is important to appreciate that this function changes with time because molecules constantly enter or leave a given volume element in this space.[98]

In Γ-space, however, the state of the gas is represented by a single point *P*. Given some arbitrary position of *P* at some time *t*, its position at any other time is determined by Hamilton's equations of motion and so, as time

[95] Post, p. 193. In other words, the *image* of the original volume is preserved by the Hamiltonian flow.

[96] Kuhn op. cit., pp. 54–7.

[97] Ehrenfests P. and T. 1912, pp. 17–39. If the molecules have *S* degrees of freedom (e.g. rotational as well as translational) then this space has $2SN$ dimensions.

[98] It can be shown that the cell volume will remain invariant in time only if certain conditions are met (see Huang 1963, pp. 57–8).

goes on, *P* describes a trajectory in Γ-space, lying entirely on a hypersurface of energy.[99] It is in the context of *this* phase space that Liouville's Theorem applies, in the sense that if *P* is contained within a 6*N*-dimensional volume *dp dq* at time *t*, then it will be contained in a Γ-space volume of the same size at all future times.

Boltzmann's justification for his particular choice of probability measure rested on the claim that if there were initially *N* molecules in a cell in μ-space, then they would all move together with the passage of time and always be contained in a cell of the same volume. This is only true if the molecules are restricted to undergoing interactions with fixed scattering centres and with zero intermolecular forces, but this is completely unrealistic. Liouville's Theorem applies, as we have just said, not to the cells in μ-space but to the cell containing the representative point specifying the state of the entire system in Γ-space. It is the volume of this cell which remains invariant in time, whereas molecules initially in the same cell in μ-space will be widely dispersed as the system evolves.

As it stands, therefore, Boltzmann's argument permits no conclusions to be drawn concerning the relative probability of the different possible spatial locations of the molecules. This is a serious deficiency in his theory as a whole which was not noticed by his contemporaries, nor was it corrected in his later 'Gas Theory'.[100] Indeed, Kuhn has suggested that Boltzmann's attribution to the cells of μ-space of the non-statistical behaviour which could not be attributed to the molecules themselves was symptomatic of his reluctance to completely relinquish deterministic dynamical considerations in general.[101] It was this switch of attention which allowed him to circumvent Loschmidt's criticism and retain something akin to his original deterministically phrased version of the *H*-Theorem. However, as Kuhn has emphasized, there is a conflation of three different notions concerning the distribution of molecules in Boltzmann's work.

The first, a 'molecular' notion, involves the precise determination of the position and velocity of each particle within a cell. The second notion applies to the distribution of molecules within cells and considers collectives of trajectories rather than the individual molecular trajectories themselves. Thus it applies to the Maxwellian *f*-function, for example, and is generally concerned with 'micro-probabilities'. The third applies to the distribution of molecules over cells, again with nothing specified about molecular positions within cells,

[99] Ehrenfests op. cit., p. 18.

[100] Boltzmann 1910.

[101] op. cit., p. 57.

and includes macro-probabilistic notions like that of the permutability measure. Boltzmann subsequently subsumed both of these latter notions under the general term of 'molar' concepts. This has led Kuhn to identify a two-way tension in the work, as Boltzmann sought to avoid the Loschmidt criticism.[102]

However, we want to distinguish the third notion from the first and second as this further illustrates the *metaphysical* difference between the *H*-Theorem and Combinatorial Approaches. The first and second notions above can be seen as underpinned by Space-Time Individuality,[103] whereas the third is not, or at least, is not *necessarily*. By eliminating time as a variable in the description of the system, the Combinatorial Approach replaces a deterministic consideration of the dynamics of the system by a process involving random choice. Thus in carrying out the calculation of the state probabilities, Boltzmann assumed that

> … the kinetic energy of each individual molecule is determined, as it were, by a lottery which is selected completely impartially from a collection of lotteries which contains all the kinetic energies that can occur in equal numbers.[104]

This is where the power of this approach lies: it is not restricted to particular molecular models for which all collision mechanics must be worked out in detail, but can be employed for any system for which the spectrum of possible energies is known. As Boltzmann wrote, this approach enabled one to determine the probability of a distribution in a way that was "… completely independent of whether or how that distribution has come about".[105] To a large extent this explains why the Combinatorial Approach was used almost exclusively in the development of quantum statistical mechanics, as we shall see.

As we have already noted, this 'independence' of the determination of the probability from considerations of the history of the system has been taken to imply that if the particles are to be regarded as individuals, then the relevant 'Principle of Individuality' must be some form of Transcendental Individuality, other than Space-Time Individuality (STI). However, this conclusion is muddied by the justification for the assignment of equal probabilities, or weights, to equal volumes of phase space, insofar as this involves an appeal to

[102] op. cit., p. 57.

[103] In the case of the second notion we might take it to be *weakly* underpinned, since it involves not the trajectories themselves but sets of them.

[104] Boltzmann 1968, Vol. II, p. 172.

[105] Ibid., p. 168.

Liouville's Theorem. The way that Boltzmann himself invokes it, in μ-space, seems to reintroduce a dynamical component based on considerations of continuity of path. This is diluted somewhat by its correct application to Γ-space where it is the history, not of individual molecules, but of the system as a whole that is considered. This is not quite tantamount to the reintroduction of the metaphysical basis of space-time individuality, since we are considering the evolution of the whole system in a multi-dimensional phase space. And, of course, even this could be avoided if one were to simply accept the weighting assignment as one of the axioms of the theory, or as justified pragmatically by the empirical results obtained on the basis of assuming it. As we shall see, the justification of these weights is deeply problematic and these latter two options have in fact been appealed to by workers in the field. This is important not only from the point of view of the physics involved, but also with regard to the metaphysical conclusions that can be drawn from this physics.

Let us return to our history. After the publication of the 1877 paper, Boltzmann again felt that he had solved the problem of the foundations of the Second Law of Thermodynamics. However, when he returned to statistical mechanics in the 1890s,[106] prompted by further criticisms, it was the *H*-Theorem approach which he deployed and further developed.[107] Here he abandoned the claim that *H* necessarily decreases and insisted that what he had proved was that if the system is in a state for which *H* is greater than its minimum value, then it is highly *probable* that it will decrease and if the situation envisaged by Loschmidt obtains, then *H* may indeed increase, before decreasing again, but the probability of such situations, whilst not zero, is extremely small. These claims were presented in the context of an analysis of the change of *H* with time, generating the so-called '*H*-curve'.

As an illustration, Boltzmann introduced a model in which black and white balls were randomly selected from an urn and then replaced.[108] In the course of the drawings, the number of white balls, for example, settles down to an 'equilibrium' figure, but occasional deviations are possible, with the chance of such a deviation being lower the greater the size of the deviation itself. The point of the model, for Boltzmann, was to act as a consistency check: it

[106] Cercignani notes that this period of Boltzmann's life is marked by restlessness, dissatisfaction and the emergence of a form of manic-depressiveness, triggered by a combination of fluctuating external circumstances concerning jobs, for example, and personal tragedies such as the death of his eleven year old son from appendicitis (1998, pp. 22–6).

[107] See Kuhn op. cit. and Cercignani op. cit., pp. 100–9 and pp. 129–33.

[108] Boltzmann 1897.

showed that assigning certain properties to the *H*-curve for gases (viewed as a continuous form of the corresponding curve generated by the ball and urn lottery) would not lead to a contradiction.[109]

What is particularly interesting for us, however, is that here there is a possible connection between the *H*-Theorem and Combinatorial Approaches. The Ehrenfests deployed a similar model in which balls are distributed between two boxes *A* and *B*, but, significantly (for us!), the balls are labelled with numbers from 1 to 2*N*. An integer between 1 and 2*N* is then chosen at random and the ball with that label is moved from its box to the other one. As the process is repeated there is a tendency for the numbers of balls in the boxes to become equal, but with fluctuations around this equilibrium value such that the probability of such a fluctuation, as measured by the *frequency* with which it occurs during the process, is lower the greater (or higher) the fluctuation is from the equilibrium value. Note, first of all, the importance of the labels: it makes a difference which ball is moved from which box. Secondly, we can view this model as a generalization of the simplistic balls-in-boxes picture that underpins contemporary discussions of particle individuality, as indicated at the beginning of this chapter. Thirdly, and relatedly, it connects the history of the Combinatorial Approach with Boltzmann's further development of the *H*-Theorem analysis by showing how the combinatorics taken over time lead to a form of the *H*-curve.

This connection is ultimately based on a deeper one concerning the meanings of the term 'probability' in each approach. On the Combinatorial side, it was taken to mean a quantity measured by volume in phase space, as we have seen; with regard to the *H*-Theorem, and certainly by the time of the discussions of the *H*-curve, it meant something measured by frequency in time. If it could be shown that these two meanings were in fact equivalent, then the conceptual gap between the two approaches could be closed. It is just such an equivalence which is established by the ergodic hypothesis which states that in the course of time a system will pass through every point on the energy hypersurface in Γ-space. The fraction of time spent by the system in some region of this space will then be proportional to the volume of that region and the two senses of probability above could be equated. The hope evinced by the Ehrenfests' model could then be realized, as the combinatorial arguments of 1877 would be reinterpretable in a kinetic manner to explain the evolution of the system in time. This would in turn provide the required

[109] Cercignani op. cit., p. 132.

justification for the claim that the Maxwellian distribution will predominate overwhelmingly in time over all other appreciably different state distributions. However, a proof of the ergodic hypothesis cannot be given, as is well known.[110] Thus the two meanings of probability, in terms of phase space volume and frequency, cannot be equated and the bridge between the Combinatorial and *H*-Theorem approaches cannot be built.

So much for history! The points we wish to emphasize are the following. First of all, classical statistical mechanics rests on the assumption that particles of the same kind, although indistinguishable, are individuals. This can be inferred from the form of statistics employed: a permutation of the particles gives rise to a new complexion which then features in the relevant counting. Secondly, Boltzmann himself made such an assumption explicit and incorporated it into the axioms of his theory of mechanics. The form, or 'principle', of individuality he presented can be seen as what we have called 'space-time individuality' and it obviously meshes nicely with the *H*-Theorem approach, particularly as originally presented in 1872. Thirdly, however, one can conceptually delineate an alternative, 'Combinatorial', approach which is not explicitly grounded on consideration of particle trajectories. Here the statistics is cashed out in terms of the distribution of particles over volumes, or cells, of phase space. It is this approach which has formed the basis of philosophical arguments to the effect that classical statistical mechanics incorporates 'Transcendental Individuality' (TI). Nevertheless, and fourthly, some justification must be given for the further underlying assumption that, on this understanding of the statistics, equal volumes of phase space have equal probability of being occupied; or, equivalently, that the boxes in the philosophers' model are all equivalent so that a particular arrangement, or complexion, is not favoured over any other. As we have seen, Boltzmann himself appealed to dynamical considerations involving not the trajectories of individual particles, but that of the system as a whole; alternatively, one could rule out such considerations entirely by taking the above assumption as one of the axioms of the theory.

[110] The impossibility of ergodic systems was demonstrated independently by Plancheral and Rosenthal in 1913. Both proofs hinged on the fact that the ergodic hypothesis implies that a multi-dimensional region of the energy hypersurface must be mapped onto a line of finite length, continuously and one to one. Rosenthal showed that this is impossible on the basis of Brouwer's proof of the invariance of dimensionality under continuous one to one mappings. Plancheral argued that such a mapping is forbidden on Lebesgue's theory of measure, since a line is a set of measure zero with respect to a region of two or more dimensions. Interest then shifted to the question whether a system could be quasi-ergodic, in the sense of passing arbitrarily close to every point on the energy hypersurface (see ter Haar 1967, p. 302 ff; Farquhar 1964; Lavis 1977).

Let us now return to the domain of the philosophers, where we shall encounter all these points again.

## 2.2 CLASSICAL STATISTICAL MECHANICS AND INDIVIDUALITY REVISITED

We recall our fundamental question: how should we understand the individuality of the particles? The metaphysically economical option is to focus on their properties and adopt a form of the Principle of Identity of Indiscernibles. However, as we indicated in Chapter 1, care must be taken to make clear *which* form we are considering. We recall that the logically weakest form, PII(1), states that it is not possible for there to exist two individuals possessing all properties and relations in common, whereas PII(2) excludes those which can be described as spatio-temporal from this set and the strongest form, PII(3), covers only monadic, non-relational properties. Those of Leibnizian persuasions would insist on adopting the strongest possible form, of course,[111] and even the average metaphysician-in-the-street would be interested in discovering which form holds in classical physics and which is violated. As we shall see, even in the latter case, the defender of PII has certain options available to her, although these come with a price attached. With that in mind, we shall consider each of these forms of PII in turn.

We recall that the validity of PII(1) is dependent upon that of the Impenetrability Assumption (IA), since if no two particles can exist at the same spatio-temporal location then their spatio-temporal properties and/or relations can serve to render them discernible and hence, on this view, ground their individuality. Now, as we have already indicated, impenetrability is a fundamental assumption of classical physics and we find it explicitly stated in a number of basic texts. Thus Newton characterized the elements of his 'proto'-theory of classical mechanics in terms of (1) impenetrability, (2) mobility and (3) the ability to excite the senses. In the *Optics* he wrote,

> … it seems to me that God in the Beginning formed Matter in solid, massy, hard, impenetrable, moveable Particles ….[112]

[111] We are not proposing any form of exegetical analysis here. As we indicated in the previous chapter, Leibniz's own view of this principle is deeply entangled with his metaphysics and ultimately, with his monadology.

[112] Newton 1952, p. 400.

Moving forward in time, from the beginning to the end of classical mechanics, we recall the fundamental axiom of Boltzmann's *Principles of Mechanics*:

> *Assumption 1.* We imagine that two different material points never occupy the same place at the same time or come infinitely close together.[113]

*If* classical statistical mechanics is understood as ultimately grounded on a consideration of atomic or molecular trajectories (and it is quite a big 'if', as we shall see), then IA must be regarded as crucial for assuring the possibility of re-identifiability and individuality. Of course, IA is not a necessary principle, but then if it is rejected—as in the case of fields, for example—some other foundation of equal strength must be sought or it might be contended that the entities concerned cannot be regarded as individuals at all.[114] In that case, it could obviously be argued that PII simply need not apply (as again might be claimed in the case of fields). Since this latter option is typically not taken up in the usual interpretations of classical particle physics, we can conclude that PII(1) is valid here. If two, apparently distinct, classical particles were to possess all their properties in common, including their spatio-temporal ones, then they could exist at the same spatial location at the same time and therefore, if IA is applicable, they must in fact be one and the same particle.

On the other hand, the fact that such particles (of the same kind or species) are indistinguishable, in the sense of possessing all non-spatio-temporal properties in common, whether IA is taken to hold or not, immediately suggests that PII(2) is violated. Nevertheless, there may be a way for the defender of Leibniz's Principle to escape this conclusion.[115] Given that the particles are individuals, but are also indistinguishable in the above sense, this individuality must be grounded on some other property, which is neither extrinsic (since more than one particle can possess such a property) yet does not feature in the set which renders them indistinguishable. Such properties may reflect, or be aspects of, some 'extra-dynamical unused structure' which is intrinsic and serves both to distinguish the particles and hence, on this Leibnizian approach, individuate them.[116]

[113] Boltzmann 1974, Vol. 1, p. 9.

[114] For point particles in three dimensions, the regions of coincidence in phase space have zero measure, so impenetrability is a reasonable assumption. We'd like to thank a referee for pointing this out.

[115] Apart, that is, from simply regarding this argument as a form of *modus tollens* against the ontology of classical physics itself!

[116] See for example, Sudarshan and Mehra 1970; de Muynck 1975.

This is clearly a desperate ploy. It is difficult to see how such properties could be regarded as on a par with such physically well grounded properties as mass and charge, for example, by virtue of the very fact that they are 'extra-dynamical'. Unlike mass and charge they are not the subject of any theory, nor are they invoked to account for the physical behaviour of the particles. Their *raison d'être* is solely metaphysical: to stand for that which confers individuality. Such a suggestion certainly has little to offer a supporter of the 'bundle' view of individuality since one could quite reasonably claim that if the substantial substratum is to be considered metaphysically redundant, then a property which is both intrinsic and 'extra-dynamical' is even more so!

If the usual intrinsic properties of a particle are regarded as non-relational, then PII(3) would have to be ruled out as well. Taking mass, charge, spin etc. as monadic properties seems to be the obvious option but again there is a possible escape route for the Leibnizian. It has been argued that whether or not a physical property is non-relational or monadic is not something that is given *a priori* and further consideration based on the physics involved may lead us to conclude that the above properties are, in some sense, relational, thus saving PII(3).[117] Thus mass, for example, has been taken to be a dispositional property, in the sense that 'having mass' is understood as shorthand for 'having the disposition to be accelerated by, or to accelerate, other objects in accordance with the physical equations involving inertia and gravity'. From such a perspective, everything that can be said or known about mass involves actual or possible relational facts and hence it must be regarded as relational.[118] If this approach is extended to the other intrinsic properties, such as charge and spin, then a case could be made for claiming that classical indistinguishable particles *cannot* be used as counter-examples to *this* form of PII.

This is an interesting proposal, but one might feel uneasy over the grouping together of 'mass' with such standard dispositionals as 'soluble' and 'fragile', for example. The source of this unease is not hard to locate. Dispositionals are standardly regarded as a particularly respectable form of subjunctive conditional[119] because they implicitly introduce into the analysis some theory of underlying structure possessed by the object concerned, by which any statement involving the dispositional can be translated into causal terms.[120] The claim that something is soluble in water, for example, can be restated in

[117] Hoy 1984.

[118] Ibid., p. 288.

[119] More respectable than 'If Caesar were in command he would use the atom bomb', to use a well-known example.

[120] See, e.g., Quine 1960, p. 223.

terms of that thing possessing a particular structure suited to dissolving or, better, which causes it to dissolve. In other words, the dispositional can be brought out of the conditional mode and translated into causal terms only because 'soluble' is reducible to some other, more basic, properties of the thing concerned.

However, it is not immediately obvious that *mass* can be regarded as reducible in this sense. Certainly, most of the standard theories in physics, whether classical or quantum, introduce it as a basic property, irreducible to any other[121] (irreducible in the sense that such a property has not been shown to be empirically reducible to any other property or collection of properties in the way that temperature has been reduced to mean molecular kinetic energy, to use a very well-known example).[122] There is, of course, the geometrodynamic programme in General Relativity, which attempts to reduce matter to regions of special curvature in space-time and the properties of matter, such as mass or even charge, to the metrical and topological properties of that space-time.[123] If such a programme were ultimately to prove successful, then the proposal that mass should be regarded as a dispositional property would carry a great deal more force. As it stands, however, in the absence of any underlying structure, mass as a dispositional cannot be translated into causal terms but must remain a simple dis-respectable conditional. Then it becomes extremely doubtful whether science should, or even can, involve properties which have the same status and carry the same force as ordinary subjunctive conditionals.

There is a further point which has been raised which runs as follows: if 'non-relational' is understood as 'independent', in the sense that it is possible for the thing possessing such a property to exist in the absence of any other thing,[124] then it is presumptuous to make any *a priori* claim for such independence/non-relationality, since the history of physics suggests that properties which were initially thought to be independent can be interpreted, within the context of a particular theory, as quite the opposite. One of the best-known examples is Mach's hypothesis, which attempts to reduce inertial mass

[121] Although in certain forms of Bohmian quantum mechanics the only property is position and mass is understood to be a feature of the quantum potential.

[122] This is not to say that producing an appropriate analysis of this notion of reducibility will be unproblematic

[123] See Misner, Thorne and Wheeler 1973 and in particular the account given in Graves 1971.

[124] Where, in true anti-substantivalist tradition, space and time are excluded from the category of 'things'.

to an interaction between bodies. This has been invoked in order to argue that mass, and by implication other intrinsic properties, can be regarded as relational.[125]

Of course, Mach's hypothesis is contentious, as is well known,[126] and, indeed, it may actually be to the benefit of the supporter of PII if it does not hold, since matter as the source of the gravitational field could be reduced to the field itself. This of course is precisely the aim of the geometrodynamic programme.[127] In this case PIIs (1) and (2) would effectively escape refutation but only because there would no longer be any separately indiscernible things to act as putative counter-examples.[128]

Nevertheless, although we have reservations about the details, we agree with the general tenor of the argument: whether a given property is understood as monadic or not cannot be decided on *a priori* grounds but only within the context of currently accepted physics. Thus it becomes a contingent affair. However, it is our belief that given this context, there is little in the way of compelling evidence to support the view that intrinsic properties are relational and therefore particles possessing such properties *can* stand as counter-examples to PII(3).

There is one final argument that we must deal with before moving on. This is to the effect that since possible worlds containing only a single object cannot be empirically investigated, science cannot *verify* the claim that a property is independent and hence monadic.[129] This a version of an old ploy often used by supporters of PII, to the effect that the visualization of possible worlds always introduces a tacit observational element by which distinguishability can be introduced into the apparently problematic situation (problematic from the point of view of PII, where such situations include universes containing two indistinguishable globes, for example, or the single object universe considered

[125] Hoy op. cit., p. 288.

[126] In particular, Mach's Principle cannot be reconciled with the de Sitter solutions of Einstein's modified field equations in which one obtains a space-time structure and hence a 'guiding' field and inertia in the complete absence of matter (see, e.g., Jammer 1969, pp. 192–8).

[127] Some care should be taken regarding the relationship between Mach's hypothesis and Wheeler's geometrodynamics. Hoy, for example, presents them as both supporting the idea that inertial mass is dependent on the gravitational effects of the rest of the universe (op. cit., pp. 288–9). However, in the Machian case inertia is determined by other bodies, whereas on Wheeler's view it is determined by the field to which matter is reduced. Furthermore, Hoy claims that the above idea implies that the "… intuition that a ball bearing could have mass though nothing else existed is wrong" (ibid., p. 289). However, it is not made clear what is meant by 'nothing else': other bodies plus space-time or other bodies only. If it is the former, then space-time counts as a thing, which would be a somewhat uneasy stance for a Leibnizian to adopt, of course; if the latter, such a claim would only be true if Mach's programme could be implemented and there are serious problems with this.

[128] We will come across a similar possibility in our discussion of quantum particles as non-individuals.

[129] Hoy ibid., p. 289.

here).[130] However, such claims make a fundamental confusion between what is involved in the process of thinking about or visualizing a state of affairs and what is entailed by the state of affairs itself. Of course, it is impossible for me to imagine a two- or one-object universe without the 'me' that is doing the imagining but it is possible for me to imagine a situation in which such a universe existed but the 'me' did not.

Indeed this is precisely the case in scientific 'visualizing' where not only single-object universes but zero-object universes are contemplated, as in the well-known solutions to the equations of General Relativity noted above. Following Leibniz, it might be objected that such speculations do not correspond to 'genuine' possibilities in some sense.[131] In the case of zero-object universes, such objections go to the heart of the debate over the nature of space-time, which we shall touch on below. To rule out single-object universes, however, requires further justification and it is difficult to see from where this might come. Obviously we cannot actually empirically investigate such a universe but the independence of a particular property may be established indirectly, via a mixture of empirical and theoretical reasons.

Similar considerations apply in the case of the following argument: since two ball bearings, say, possess different dispositions to affect a test particle placed in their vicinity, these can count as distinguishing relational properties in a two-bearing universe.[132] Clearly the introduction of a 'test particle' is equivalent to the introduction of an observational element by which the two things may be distinguished. If one is considering a possible world containing two ball bearings *only*, then there is no test particle for them to affect and thus no way of distinguishing them. Standardly, dispositions are simply not taken to be possessed by objects in the same way that intrinsic properties are said to be, since they are merely a form of subjunctive conditional, as we have noted, and are only *realized* when some extraneous factor is introduced. The disposition to affect a test particle is only realized and the test particle only actually affected, when it is actually introduced into the situation! Thus the

[130] See, e.g., Ayer 1954, p. 33 or Casullo 1982.

[131] Thus, Mach objected to Newton's bucket experiment on the grounds that it did not correspond to a genuine possibility since it left out precisely that which was most important, namely the fixed stars. More recently, as we have already noted, Hacking has responded to the well-known universe-with-only-two-globes counter-examples to PII by arguing that such situations do not count as genuine possibilities as they stand because they leave out the spatio-temporal background. Leaving such considerations aside, there is a fundamental issue here, to do with the very basis of our claims of possibility and the establishment of such a basis in a non-question-begging way. We shall adopt a broadly naturalistic approach in insisting that any theory of possibility which does not admit scientific 'visualizations' and speculations should be rejected.

[132] Hoy op. cit., p. 291.

ball bearings can only be distinguished when an extra-theoretical element is included, but that is precisely what is ruled out by our initial premise.[133]

Hence we see that the question is not simply one of whether PII can be retained or not, but whether the price of that retention is too high. PII(1) comes cheap, as most commentators would accept the validity of IA in the classical domain. PIIs (2) and (3), however, seem too expensive, metaphysically speaking, since their price involves certain views of intrinsic properties which are highly problematic, to say the least.

Let us now turn to alternative accounts of individuality and see how they fare in classical physics.

We have already noted that impenetrability, together with continuity of trajectory, are the two vital components of Space-Time Individuality (or STI). The latter is, of course, expressed in the classical equations of motion governing the particle. In Hamiltonian form, these can be written as follows:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}; \frac{dp}{dt} = -\frac{\partial H}{\partial q} \tag{2.2.1}$$

where $t$ is the time, $H$ the 'Hamiltonian', $q$ the generalized coordinate and $p$ the conjugate generalized momentum. The uniqueness of the solution of these equations of motion, given a set of initial conditions, then guarantees the re-identifiability and individuality of the particle through space-time and, in particular, through a permutation.[134] What underlies this approach is the view that even in the absence of an intrinsic difference in the properties of the particles—that is, even if the particles are indistinguishable—the extrinsic difference of kinematical behaviour is a sufficient condition for their individuality and trans-temporal re-identifiability. This derives its operational meaning

[133] Of course, on a different view of dispositions, this argument may not hold. If one were to regard them as primitive or, minimally, as non-reducible to non-dispositional properties in some sense, then one could insist that they could serve to distinguish. But of course, one must still be careful: it is not enough to note that if we had a steel ball bearing and a glass ball bearing, these objects can differ with regard to the dispositional property of fragility, even if that disposition is never realized. In this case, the ball bearings are already distinguishable in terms of their non-dispositional properties, having to do with their composition. And finally, this sort of general concern further underscores the point that whether PII is violated or not depends on further considerations which are ultimately metaphysical in nature.

[134] "… two particles, once 'told apart', can always be 'told apart', for they can always be re-identified, thanks to the uniqueness of the solutions of the equation of motion." (Jammer 1966, pp. 341–2).

from the assumption of the unrestricted possibility of establishing an unbroken connection between the object at time $t_1$ and the object at time $t_2$, "… by continuous observation (either direct or indirect) through all intermediate time".[135]

Pais gives the following illustration:[136]

> Suppose I show someone two identical balls lying on a table. Next I ask him to close his eyes and a few moments later to open them again. I then ask him whether or not I have meanwhile exchanged the two balls. He cannot tell, since the balls are identical. Yet I do know the answer. If I have exchanged the balls then I have been able to follow the continuous motion which brought the balls from the initial to the final configuration.[137]

In the context of particle permutations, Tolman has likewise noted that

> … our procedure in regarding the interchange of two similar molecules as corresponding to a significant change in the mechanical state of a system, even though not in its condition, evidently implies the possibility of keeping a continuous observation on the system which would let us know whether two similar molecules change roles or not. This, however, is in entire agreement with the point of view of the *classical mechanics*, which would permit such a continuous observation, at least in principle ….[138]

However, is continuity alone enough to allow us to uniquely label and individuate the particles? What about the first condition in the above characterization of Space-Time Individuality, concerning impenetrability? Insofar as this is historically associated with the *substantial* aspect of matter, including this in one's characterization of STI might be seen as actually undermining this position. Continuity on its own is not sufficient to guarantee individuality, and hence re-identifiability, on this view, since if two space-time trajectories were to cross, one would not be able to tell which particle was which and the above possibility of following the continuous motion would be lost. But then what guarantees impenetrability? Why should the points of space-time be monogamous or virginal, as Quinton put it? One response would be to insist that impenetrability is of the *nature* of substance but in that case the

[135] Bridgman 1927, p. 92.

[136] This is the argument he gave to one of the authors (French) at the *International Conference on Ludwig Boltzmann*, Vienna, 4–7 September 1981, where elements of this chapter were first presented as part of a paper entitled 'Boltzmann Statistics and Individuality'. Despite some considerable discussion, Pais (and others who were present) remained unconvinced that Maxwell-Boltzmann statistics could be understood in terms of non-spatio-temporal TI.

[137] Pais 1979, p. 893.

[138] Tolman 1962, p. 77.

re-identifiability of the particles, through permutations in particular, is being underpinned by something non-spatio-temporal.

This combination of spatio-temporal with non-spatio-temporal aspects can be found in Reichenbach's account of trans-temporal identity in terms of the notion of 'material genidentity'.[139] 'Genidentity' in general is the relation which connects different states of the same thing at different times. According to Reichenbach, it is the properties of this relation which our notion of physical identity is based upon. *Material* genidentity is defined in terms of three characteristics:

(1) continuity;[140]
(2) impenetrability, which Reichenbach regards as a 'characteristic of material objects';[141]
(3) the possibility of labelling, in the following sense:

> … we find that whenever two material objects exchange their spatial positions this fact is noticeable. We usually recognise this change of position by the use of specific marks on the objects … These marks remain on the object in accordance with the continuity criterion and permit an identification of the objects even when no observation during the change of spatial positions was made and the continuity criterion cannot be applied. In other words, an interchange of spatial positions is a verifiable change even though no records of the act of interchanging are available.[142]

These characteristics are necessary but not sufficient, since, for example, if a house is knocked down, the resulting heap of rubble is not called the same thing as the original house, even though all three criteria are satisfied. Here the issue is one of relative identity and the three characteristics need to be supplemented with some kind of sortal concept which might cover both 'house' and 'heap of rubble'.

[139] Reichenbach 1956, p. 38 and pp. 255ff.

[140] Ibid., p. 225.

[141] Ibid.

[142] Ibid. There are two things to note about this passage: first, Reichenbach's empiricism comes to the fore in the emphasis on 'verifiable change'; secondly, the idea of 'marking' an object turns out to be central to Reichenbach's account of causality, since causes are distinguished from their effects, independently of their temporal order, through the introduction of a small variation, or mark: if $e$ is the cause of $e'$, then a mark on $e$ is associated with a mark on $e'$ but not vice versa (1958, p. 136). This 'mark method' is absolutely crucial for Reichenbach's 'causal' theory of time. What is interesting for our purposes is the relationship between genidentity and causality: whereas Reichenbach insisted that different events can be genidentical only if they are causally related, Grünbaum reversed this relationship and argued that causality was grounded on genidentity, taken as primitive (1967, p. 57). For further discussion see Hoy 1975.

The third characteristic, in particular, is crucial since it is this that is 'translated into a statistical property' in the context of classical statistical mechanics, by supplying the *definition* of an arrangement, or 'complexion', of particles over states/ balls over boxes.[143] Furthermore, Reichenbach insists that experimental tests of the laws of statistical mechanics (and, in particular, those that effect the reduction of the thermodynamical notion of entropy) constitute a test for the assignment of material genidentity to the particles because such laws are ultimately based on the Maxwell-Boltzmann distribution formula which in turn is based on this definition of arrangement (that is, one in which a permutation of the particles counts). Such tests, he claims, have been positive in the realm of classical physics, but negative in quantum mechanics where, he argues, material genidentity must be replaced by 'functional genidentity', in which characteristics (2) and (3) above may be violated.

However, such claims are problematic. What the experiments provide is a test of characteristic (3)—that the particles can be labelled and their permutation is regarded as having physical significance—rather than of material genidentity as a whole. Although Reichenbach associates this characteristic with the continuity of trajectory expressed in criterion (1), his view of impenetrability, insofar as it is taken to be a characteristic of material objects, can be regarded as involving non-spatio-temporal notions.

This view finds a home in an account which understands individuality to reside in something over and above both the intrinsic properties of the particles—in terms of which they can be regarded as indistinguishable—and also their spatio-temporal properties. Thus it ascribes a form of 'Transcendental Individuality' (or TI) to the particles. As we indicated in Chapter 1, such an account can be cashed out in various well-known ways: in terms of some kind of underlying Lockean substance,[144] for example, or in terms of primitive thisness,[145] or more generally, one might approach it in modal fashion, through the doctrine of haecceitism, which—we recall—asserts that two possible worlds may describe some individual in qualitatively the same way, yet differ in their *de re* representation of that individual.[146] We shall lump these together for now under the rather clumsy heading of 'non-spatio-temporal forms of TI'.

[143] Reichenbach 1956, p. 229.

[144] French 1989a.

[145] Teller 1995. One might be tempted to claim that 'primitive thisnesses' are impenetrable, but then if they have such properties, one might wonder whether they can still be called 'primitive'!

[146] Lewis 1986; Huggett 1999.

We recall that STI is more economical, metaphysically speaking, in the sense that it claims that it is the points of space-time which *confer* distinguishability, re-identifiability and individuality. Proponents of the non-spatio-temporal alternatives, on the other hand, insist that these points have only a secondary role to play in the sense that they allow us—via the trajectories—to *infer* individuality but that it is something else, the Lockean substantial substratum, or some privileged property of thisness, which, ontologically speaking, *confers* it.

Now, the question is: does the physics itself give us grounds for preferring one of these alternatives over the other?

It has been suggested that classical statistical mechanics, understood primarily in terms of what we have called the Combinatorial Approach, implies non-spatio-temporal TI. Thus Post writes that, "Boltzmann, who founded statistical mechanics at the end of the last century, extended this notion of transcendental individuality implicitly to the atomic realm".[147] This suggestion has been reinforced by the form of argument given at the beginning of this chapter where one simply presents the distribution of balls over boxes without thought or consideration of the underlying spatio-temporal trajectories. However, our historical digression should give us pause at this point: we have noted the underlying assignment of equal *a priori* probabilities to the different complexions, including, crucially, those which differ by a particle permutation only. One possible justification of such an assignment would be to adopt some sort of ergodic approach to underpin the claim that the system occupies every possible state with equal probability. However, as we have indicated, such approaches typically involve consideration of the time evolution of the system (albeit in Γ-space). Thus, if one were to attempt to justify the assignment of equal *a priori* weights through ergodic considerations, then TI, in the above sense, would have to be supplemented with some spatio-temporal elements.

The alternatives are to insist that such an assignment is justified by its well-confirmed empirical consequences[148] or to simply include the assumption of equal *a priori* probabilities in the axioms of the theory.[149] In this case a non-spatio-temporal form of TI would indeed be sufficient for classical statistical mechanics, but it can be shown that it is not *necessary*. In order

[147] op. cit., p. 15.

[148] This is the option taken up by many physicists, in fact.

[149] As we shall see, it has been argued that this assumption should be abandoned in the case of quantum statistics in order to allow the restoration of a classical ontology of individuals.

to demonstrate this, we need to establish STI as a viable alternative. As we have indicated, the sorts of argument given by Pais and others above are not quite up to the job, since they leave open the possibility that impenetrability must be understood in substantial terms.[150] What is required is an alternative understanding in which such substantial considerations are absent. Surprisingly, just such an understanding can be found in the unpublished papers of Newton[151] who indicated how particle mechanics could be re-written in terms of fields, with the notion of 'particle' effectively reduced to the characteristics of impenetrability, 'mobility' and the 'ability to excite the senses'. We shall now outline this alternative 'field-theoretic' approach. Insofar as it demonstrates the full viability of STI in the classical context, it leaves us with an interesting situation: contra Reichenbach, the physics is compatible with two different metaphysical packages regarding individuality.

## 2.3 A FIELD-THEORETIC APPROACH

As well as particle theories such as statistical mechanics, classical physics also encompasses field theories such as electrodynamics. The distinction between the two kinds of theory is most obvious when one considers the question of matter and its interactions. The standard view holds that matter is composed of particles which *interact* via fields of force. Various alternatives have, of course, been propounded and vigorously supported, such as the Boscovichean 'field' theory of matter and the 'action at a distance' view of interactions (usually associated with a 'pure' particle theory of matter),[152] and we acknowledge that the distinction as expressed here is a crude one.

As it stands it is clearly linked to a difference in the underlying ontologies of the theories; in particular as regards their fundamental individuals. In field theories, certain properties are associated with every point of space-time. There are then (at least) two ontologies which are possible. One is to argue that the

[150] Of course, if we consider the time evolution of the system in phase space then impenetrability is not required (at least, not explicitly) since trajectories never cross in this space—they are either always distinct or coincident. However, we must recall the construction of such a space: if momentum is defined as mass multiplied by the time derivative of the particle's position, then re-identifiability of the particle at all times is presupposed. In other words, distinct space-time trajectories are implicitly assumed in the construction of the space. One could, of course, choose a different definition of momentum but that is problematic; we are grateful to a referee for drawing our attention to this issue.

[151] See Hall and Hall 1962, pp. 138–40; also Koslow 1976.

[152] See, for example, Hesse 1961.

field is ontologically independent of these points, in much the same way as particles can be said to be, but with some correspondence existing between the points and the field quantities. The other is to regard the field quantities as properties of the space-time points themselves.

Taking the first possibility, fields are then granted the status of independent, real entities which, furthermore, may be regarded as particulars. The fact that they are 'global' particulars, in the sense of not being localized to a particular region or point of space at a given time, is of course not an issue since the spatial extent of some thing is just another property of that thing and cannot preclude it from being regarded as an individual on this view (in this case, size really doesn't matter!).[153]

According to this view, then, a field is an individual and can be given a label, or proper name, which designates the underlying substance, for example, or the privileged property of 'thisness'. With regard to the first option, it may be argued that one of the most striking developments of late nineteenth- and early twentieth-century physics was the emancipation of the field concept from the notion of material substance.[154] However, such arguments tend to involve a form of essentialism in the sense of utilizing a possible answer to the question 'what is the essence of substance?' in order to justify the exclusion of fields from this category.[155] Thus, one such answer involves impenetrability and since fields are not impenetrable, they cannot be substances. This is true, but although impenetrability certainly featured in the characterizations of substance of Descartes and Newton, as we noted above,[156] by the time we get to Maxwell, we can see the development of a dynamical conception in which substance is regarded as the recipient of momentum and energy.[157] Einstein has also been taken as supporting such a view via his famous equating of mass and energy through $E = mc^2$. Furthermore, it has been argued that the energy of a system can be used to re-identify that system[158] and that energy therefore provides the fixed point of reference through change and which is necessary for the world to be comprehended.[159] Since invariance through change is a fundamental characteristic of substance, energy should be regarded as substantial.

[153] Likewise, samples of the chemical elements can also be regarded as individuals; the distinction between atoms and elements would then lie along the fault line of continuity/discontinuity rather than non-particularity/particularity.

[154] Harman 1982.

[155] We owe this point to Michael Redhead.

[156] Hall and Hall op. cit., pp. 132–44.

[157] Maxwell 1873, Vol. II, p. 181.

[158] Fowler 1926; Theobald 1966.

[159] Mach 1942, p. 607.

This response is problematic, however. Relativity theory implies that energy is a conserved quantity only within a given frame of reference and is not itself universally conserved under general coordinate transformations.[160] Although the energy of a system may not vary with time with respect to any chosen frame of reference, it will vary if the reference frame is changed.[161] This pushes against regarding energy as substantial, at least in the sense taken above.[162] The energy of a system cannot always serve to re-identify it because a change of reference frame will produce a change in the observed energy value of the system. Furthermore, it is not the case that Einstein's famous equation represents the acquisition of a substantial nature by energy. What relativity theory requires is that a change in energy corresponds to a *change* in mass, and vice versa. It does not demand that mass and energy possess the same set of properties and clearly they do not.[163]

Of course, one could always take the second option and insist that fields possess primitive thisness. Involving a primitive notion, as it does, there's not a lot that could be said against such a view on the basis of physical considerations such as the above. There's perhaps not a lot that can be said for it either, except that if thisness is understood in terms of self-identity, then a field could be understood as possessing this unique property. But here the issue of metaphysical profligacy looms. The attribution of thisness and indeed non-spatio-temporal individuality in general is redundant, since, like the angels, each field is a kind in itself, individuated in terms of its distinct properties. Thus the properties of the electromagnetic field both distinguish it from the gravitational field and confer individuality upon it. In this case, since the possibility of fields possessing the same 'intrinsic' properties and hence being indistinguishable does not even arise—so that the Identity of Indiscernibles is satisfied, albeit in an uninteresting manner—there is no need to have recourse to any form of TI.[164]

[160] As far as Lorentz covariance is concerned, the true invariant is a function of energy and momentum.

[161] In particular, energy is not an invariant of the electromagnetic field. In this case the invariant quantities are scalar quantities such as $E.H$ and $E^2 - H^2$, where $E$ and $H$ are the electric and magnetic vectors respectively, obtained from the 6-component antisymmetric field tensor.

[162] Bridgman 1961, Ch. 1.

[163] Thus energy can be viewed as a measure of the 'ability' of matter—regarded as substantial—to do work and undergo change, whereas mass is a measure of the inertia of matter and thus of matter's ability to resist change.

[164] Although, as has been pointed out to us, it is certainly possible that we might discover that a field is apparently the sum of two solutions to a non-linear field equation and hence not a solution to the equation itself. Given such a case we might well be tempted to accept that there are two indiscernible fields in this situation and an attribution of primitive thisness or TI in general might be appropriate.

The alternative is to eschew all talk of fields as individuals entirely and take them to be merely *properties* of the points of space-time. On this interpretation it is the *space-time points* which are the primary individuals and which stand in the same relation to the fields as a subject does to its predicates. This shifts the metaphysical focus to the ontological status of these points and, in particular, the issue of whether they can be regarded as individuals. We shall return to this in our discussion of space-time individuality below.

Whatever the ontological status of fields, it turns out that this distinction between particle and field theories does not uniquely and rigidly classify the subject matter of classical physics since it can be shown that classical particle mechanics can, in fact, be rewritten in field-theoretic terms.[165] Each particle is represented by a dichotomic field $\psi(r,t)$ which can possess only two values: {Yes, No}. 'Yes' indicates that a particle is present at that particular space-time location and the value 'No' indicates that it is not. Impenetrability is also assumed, so that the fields associated with two different particles cannot have the value 'Yes' at the same space-time point. Now a field is only completely specified if the field amplitude is given for *all* space-time points (unlike the particle case). Hence, if the field has the value 'Yes' at some point, it must be specified to have the value 'No' everywhere else. A particle will therefore be represented by an infinitesimally thin, sharp spike in the field and its motion will be translated into motion of this spike in the field.[166]

For simplicity's sake, let us consider motion in one dimension, labelled $x$. We can then construct a kind of 'blip space' which is represented graphically by taking the $x$-axis as the real number line and the $y$-axis as having two values only, namely 'Yes' and 'No'. The dichotomic field can now be represented by a mathematical function $f(x)$ whose domain is the real numbers and whose range is the two member set {Yes, No}; that is, $f$: $\Re \rightarrow$ {Yes, No}. In the particle description, motion is represented in terms of a particle at time $t_1$ being at location $x_1$ and moving at time $t_2$ to $x_2$. In terms of the dichotomic field in 'blip space', this can be represented as a change in field configuration from the spike at $x_1$ to the spike at $x_2$, as shown in Fig. 2.2. In this manner, classical particle mechanics can be rewritten in field-theoretic terms.[167]

[165] Redhead 1983, section 2.

[166] Huggett and Weingard 1994 argue that what we have here is not really a field as such, since it only has finitely many degrees of freedom.

[167] Interestingly, one can even introduce the Fock space of Quantum Field Theory (QFT) into classical particles mechanics and describe the motion of particles in terms of creation and annihilation operators operating on the spikes of the dichotomic field (Redhead op. cit., sections 3 and 4). We shall return to discuss this in more detail in the context of QFT and the issue of the individuality of 'quanta'.

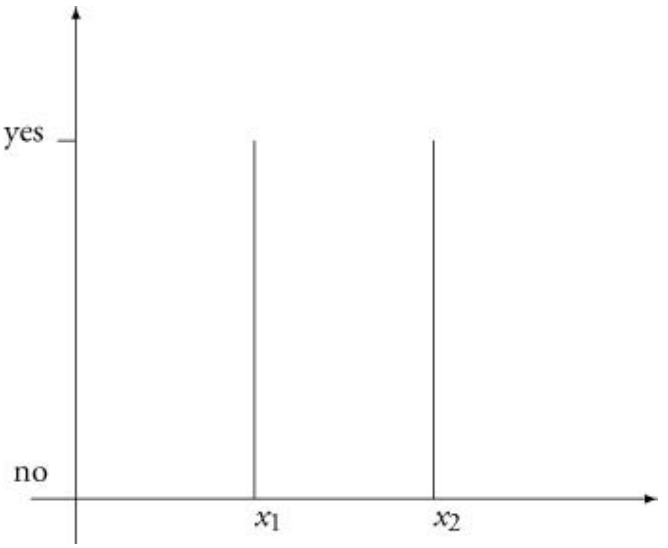


Figure 2.2. The motion of a particle in the field description.

Let us now return, once more, to our philosophically illuminating model of two particles distributed over two boxes. We recall from our historical digression that a justification of the weights given to the various arrangements, or complexions, of balls in boxes, can be attempted by employing some form of ergodic approach together with Liouville's Theorem. In this case, non-spatio-temporal forms of TI must be supplemented with dynamical notions involving spatio-temporal continuity of path in phase space. The alternative would be to insist that the weighting assignments are simply part of the axioms of the theory, with their justification ultimately cashed out in terms of the empirical consequences. In this case, non-spatio-temporal TI would be *sufficient* for classical statistical mechanics but it turns out that it is not *necessary*.

We can show this by rewriting the whole theory in field-theoretic terms and representing the particles as dichotomic 'Yes–No' fields as we have just indicated. A two-particle system would then be represented as two 'blips' in 'blip space' which move around in this space as the particles move. Here there is no underlying substance or primitive thisness (since there are no particulars), merely a property—such as impenetrability, for example—passing from one location to another.

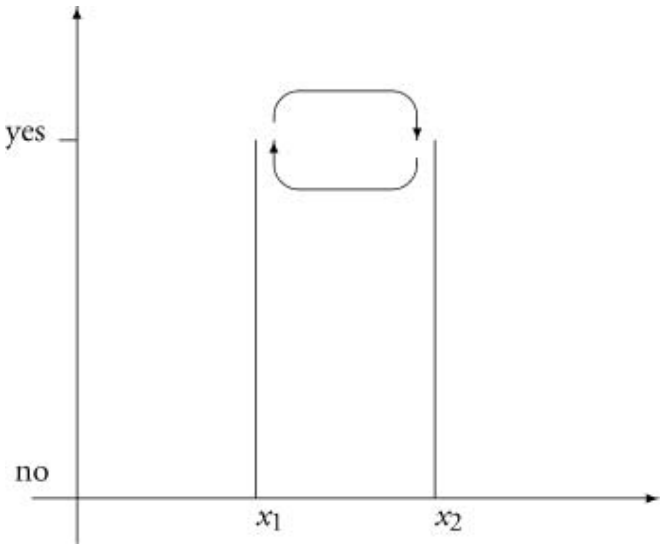


Figure 2.3. The permutation in the 'blip space'.

There is a problem with this representation, however: how can such a 'blip' be re-identified as the same blip through time? We recall from Chapter 1 that *a* and *b* are regarded as being identical in the relative sense if a at time $t_1$ is the same *F* as *b* at time $t_2$, for some sortal concept *F*. In this case, the term 'blip' is acting as the sortal concept and the grounds for referring to the 'same' blip at $t_1$ and $t_2$ rest on the continuous well-defined spatio-temporal trajectory of such a blip.

Similarly, in our two-particle case, if the two blips are permuted then the change in the situation is revealed through the change in the blips' trajectories in space-time. In our 'blip space' the permutation can be represented as shown in Fig. 2.3, where we have to introduce three dimensions in order to satisfy the impenetrability assumption and allow the particles/blips to actually move around one another. Here it is their spatio-temporal trajectories which confer individuality upon the blips.[168]

'Blip space' can be thought of as a form of 'occupation number space', in which one associates with each energy state, or box, the number of particles in that state/box in the particle approach, or one associates with each space-time

[168] It has been suggested that this is not the case if distinct dichotomic fields are associated with each particle, since—we presume—the individuality of the particles would then be grounded in the distinctness of the fields. But it is difficult to see how this distinctness could be cashed out, given that the field only has the values {Yes, No}.

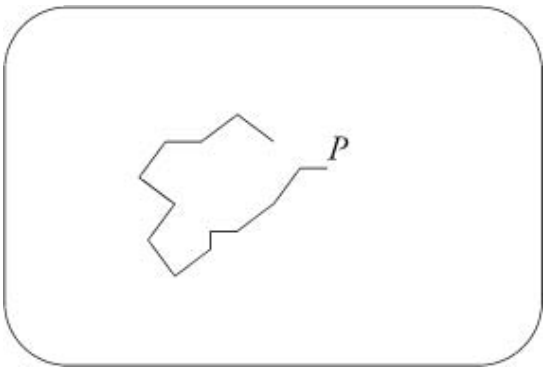


Figure 2.4. The movement of the point *P*.

point a value—0 or 1, say—according to whether it is bearing the field property or not. As the system undergoes change, its representative point moves around this occupation number space, which will be multi-dimensional in general, covering the various points available to it, as represented in Fig. 2.4.[169]

Now, there are two important points to note about such a representation. The first is that it allows us to recast classical physics in such a manner as to highlight the commonalities with quantum physics. In this way we can avoid getting tied up with *apparent* metaphysical differences that actually have more to do with the representation than the entities being considered. The second point is that—setting aside the field-theoretic considerations with which we began this discussion and just taking occupation number space on its own—all the considerations of particle permutations and TI appear to have been jettisoned. From the perspective of this representation we are not concerned with issues to do with which particles are in which state but only with how *many* particles are in which state. This shift in concern has been taken to have a profound impact on the apparent support for haecceitism provided by classical physics.[170]

We recall that *haecceitism* underpins the truth of modal statements by offering an account of individuality articulated in terms of a difference in the *de re* representation of the individual concerned.[171] If we consider our balls and

[169] Redhead represents this space in terms of his 'function space' formalism for representing theories in general (1975, p. 77).

[170] Huggett 1999.

[171] Lewis 1986; Huggett op. cit.

Figure 2.5. The arrangement (3).

boxes model again, then the different arrangements that can be obtained, corresponding to different possible states of the system, can be seen as different possible worlds. And, "… the structure in which states are represented by a theory is the structure of the universe of possible worlds postulated by the theory".[172] The question then is, given such an understanding, is haecceitism *necessary* for classical statistical mechanics? It seems not.

Recalling the distinction between Γ-space and occupation number space above, haecceitism seems to hold in the former but not in the latter. It holds in Γ-space because the two complexions corresponding to the arrangement of Fig. 2.5 are regarded as distinct, even though they are qualitatively indistinguishable, and this distinctness is due to the difference in which the permutations are regarded. This amounts to a difference in *de re* representation.[173] In occupation number space, however, we are not concerned with which particles are in which states but only in how many particles are in a given state. From this perspective, there aren't two complexions at all, but just arrangement (3) (see Fig. 2.5) corresponding to a single possible world.[174] Haecceistic differences are then simply not possible!

However, before we can conclude that we can do classical statistical mechanics without assuming haecceitism and, more generally, that we can always choose the above field-theoretic representation, we need to show that the phase space and occupation number space representations are relevantly equivalent. In the context of the argument with which we began this chapter, the question immediately arises: from the perspective of occupation number space, how do we get the appropriate *weights* required for classical statistical mechanics? From the particle perspective, the above arrangement is given the weight of two because it can be achieved in two ways, obtained by permuting the particles. According to the defender of a non-spatio-temporal view of TI, these two ways are ontologically distinguished in terms of the different underlying

[172] Huggett op. cit., pp. 8–9.

[173] Huggett ibid., p. 13.

[174] As Huggett acknowledges, this conclusion requires the assumption of the Identity of Indiscernibles as applied to *complexions*, or possible states (op. cit., p. 13). A similar assumption is made with respect to 'Leibniz equivalence' in the discussion of the hole argument, as we shall see.

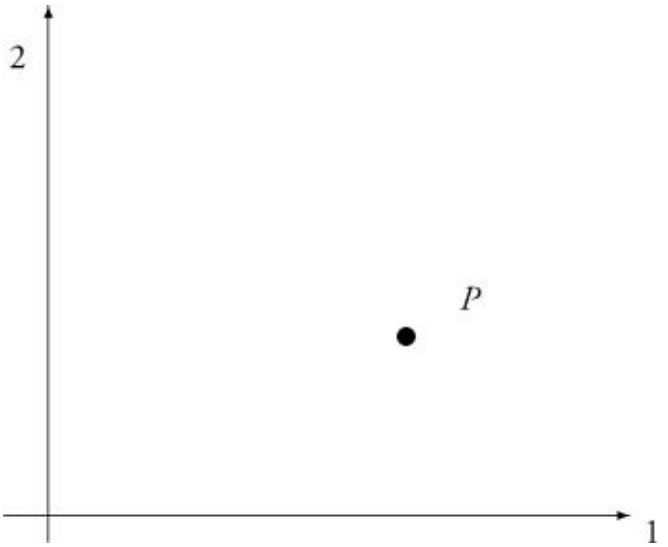


Figure 2.6. The two blip system in a two-dimensional phase space.

substances or primitive thisnesses involved.[175] In the field-theoretic approach the two complexions obtained via a permutation are not distinguished, since in both cases there is just one field property at one point, and another at another point. How, then, can the correct weights be obtained?

There are two ways of addressing this question. The first is to transform from the 'occupation number space' representation of the system to (yet) another space in which the weights can be discovered without having to resort to non-spatio-temporal forms of TI at all. This other space is an *r*-dimensional phase space, where *r* is the number of degrees of freedom of the system. Thus, considering more than just two states, our two-blip system can be represented by a point in a two-dimensional phase space as sketched in Fig. 2.6.

Again, the point *P* representing the system will move around this phase space as the system changes. We can draw a line at 45° through this space to represent the set of points where our blips pass each other as they move about.

[175] Of course, an opponent of this view might well point out that it is of the nature of both substance and primitive thisness that one 'bit' of substance or one instantiation of primitive thisness cannot be distinguished from another, in terms of substance or primitive thisness themselves, respectively, nor, obviously in terms of properties, but only in terms of one being 'here', say, and the other being 'there'. In which case, why not save all the bother and shift to space-time individuality to begin with?! This, of course, is just to recapitulate the well-known metaphysical objections to both substance and primitive thisness.

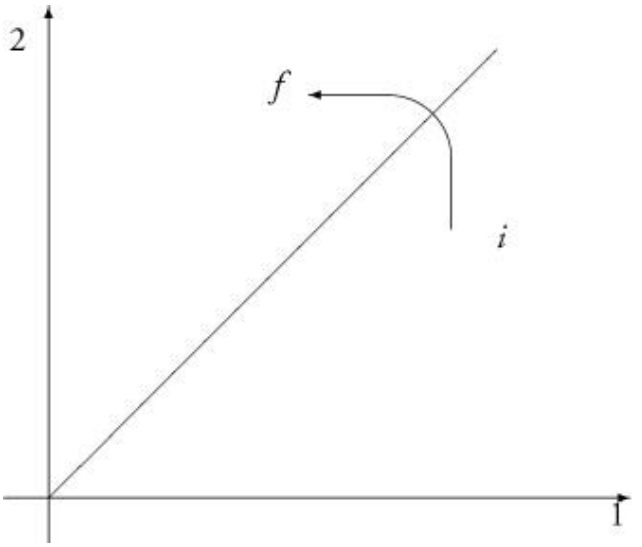


Figure 2.7. The permutation of two blips.

For any point *i*, there will be a mirror image point *f*, the reflection of *i* through this 45° line. The permutation of two blips is then described by the movement of *i* to *f* as in Fig. 2.7.[176]

Thus, in such a phase space, a permutation of the blips can be represented and distinguished by a continuous trajectory which allows the permuted complexions to be counted as distinct without requiring any non-spatio-temporal form of TI. With regard to such a permutation, when one makes this transformation, the single point in occupation number space is effectively 'exploded' into the two points *i* and *f*, as represented in Fig. 2.8. Of course, in the case of *n* particles, the point *P* in occupation number space will transform into n! i- and *f*-type points in phase space.[177]

[176] Again, a third dimension must be introduced if impenetrability is assumed.

[177] It might be objected that we haven't actually jettisoned non-spatio-temporal forms of TI at all since the relevant points of the phase space correspond to permutations and thus the separate counting of such permutations is effectively encoded in the formalism. However, this is to beg the question. The *r*-dimensional phase space is just a mathematical space and, from this spatio-temporal perspective, the relevant points should only be identified as corresponding to permutations on the basis of the movement of the blips through the space, on a continuous trajectory. In other words, it is only on the basis of a broadly spatio-temporal understanding that the points can be represented as corresponding to permutations in the first place.

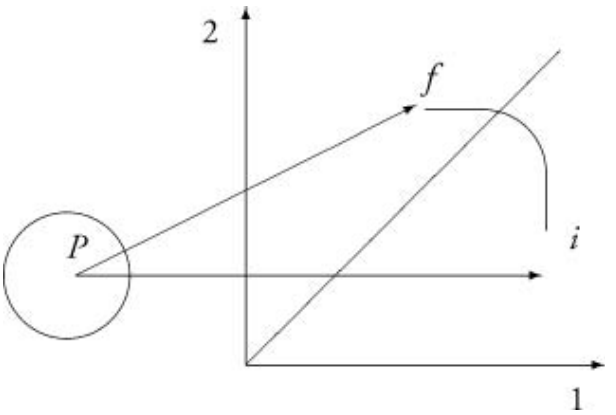


Figure 2.8. The single point 'exploded' into two points.

Alternatively, we can adopt the following strategy.[178] Let us recall Boltzmann's central expression for the permutability:

$$\mathcal{P} = \frac{n!}{\omega_0!\omega_1!\ldots} \tag{2.3.1}$$

where the $\omega_i$'s are the occupation numbers. If it is assumed that $\omega_i \leq 1$,[179] then we straightforwardly obtain

$$\mathcal{P} = n! \tag{2.3.2}$$

In other words, as we noted above, our single point in occupation number space explodes out into $n!$ points in phase space. Now, if there are $Z$ points in occupation number space, say, this will correspond to $n!Z$ points in the corresponding phase space. From the perspective of occupation number space, the probability of obtaining a given arrangement, or distribution (corresponding to (3) above), is simply

$$P_0 = \frac{1}{Z} \tag{2.3.3}$$

From the perspective of phase space, this probability is given by

$$P_p = \frac{n!}{N} \tag{2.3.4}$$

[178] Huggett 1999.

[179] Ibid., p. 16.

where $N$ is the total number of possible arrangements. But as we have just seen, $N = n!Z$; thus

$$P_0 = P_p \tag{2.3.5}$$

In other words, the statistics as viewed from the perspectives of occupation number space and phase space are *identical*.[180]

What are the philosophical implications of this result? Recall the point above that Γ-space supports haecceistic differences, whereas occupation number space does not. What the identity of the statistics shows is that haecceitism is not *necessary* for classical statistical mechanics as we can get the statistics we need on the basis of a representation that does not assume it. As Huggett puts it:

> The philosophical pay-off is that statistics do not select one space over the other and hence will not yield conclusions about modal metaphysics.[181]

Of course, this result is established on the basis of the above assumption that $\omega_i \leq 1$. How is such an assumption justified? One possibility is to appeal to our old friend, the Impenetrability Assumption:

> The key feature … which ensures the agreement of [the phase space and occupation number space representations] is … that no two Newtonian particles ever simultaneously occupy the same point of μ-space.[182]

In that case, each such point will either be occupied by one particle, or it will be occupied by none (it will be either monogamous or virginal). More generally, given a finite number of atoms and with position and momentum

[180] op. cit., p. 16. As Huggett goes on to point out, the empirical adequacy of classical statistical mechanics—understood, at least in part, in terms of recovering the empirical evidence of thermodynamics—hinges on Boltzmann's expression for the permutability, and not necessarily on any explicit counting of complexions obtained via permutations. Typically what physicists employ is a 'coarse-graining' procedure in which the cells of μ-space are grouped into blocks and Huggett shows that under such a procedure, the number of fine-grained arrangements corresponding to a coarse-grained one is equal from either the occupation number space or phase space perspectives (except for a constant that is effectively washed out in the relevant expressions for the probabilities; ibid., pp. 18–19).

[181] Ibid., pp. 19–20.

[182] Ibid., p. 16.

represented classically in terms of a number line with an infinite number of points, the assumption will also hold.[183] The metaphysical relevance of these considerations is, of course, that they are ultimately *spatio-temporal*. Impenetrability is explicitly built into STI, as we have seen, but even the less problematic justification hinges on an implicit spatio-temporal representation in terms of space-time points. This is significant, not only for our project as a whole, but also for the issue, noted previously, of the metaphysical utility of haecceitism. That this is not necessary for classical statistical mechanics does not, of course, imply that the particles should not, or cannot, be regarded as individuals.[184] Having established the above conclusion, one needs to pursue the metaphysics further and make clear on *what it is* that the individuality of the particles is grounded. And by so doing one effectively spells out the terms of the *de re* representation as we indicated in Chapter 1.

If such a spelling out encompasses non-spatio-temporal forms of TI, involving substances or primitive thisnesses for example, then haecceitism—*understood in these terms*—is certainly not necessary because one can 'do' the statistics by means of the occupation number space representation. However, insofar as this requires spatio-temporal considerations, whether implicitly or explicitly, the anti-haecceitism conclusion only follows if the spelling out of *de re* representation is *not* taken to embrace such considerations. If, alternatively, one were to insist on STI as underpinning trans-world identity of individuals, then a form of haecceitism could be resurrected. The upshot, then, is that casting the discussion in terms of haecceitism is not terribly useful from a metaphysical point of view; what does all the work in taking us to the conclusion regarding particle statistics are the considerations of TI and STI and it is on them that we should be focusing our attention.[185]

[183] Huggett, pp. 16–17.

[184] As we shall discuss shortly, it is here that the difference between classical and quantum physics is typically drawn. At the conclusion of his paper, Huggett insists that his analysis "… has been directed at showing that there are no very heavy metaphysical implications of classical physics, and that therefore anticipated innovations in the notion of an individual in quantum mechanics will not be innovative at all." (ibid., pp. 23–4) Although we agree that classical physics does not determine the particular form of the metaphysics of individuality and that—as we shall argue—one can recover quantum statistics with a metaphysics of individuality even in this domain (albeit with significant differences from the classical form), nevertheless we maintain that *some* such metaphysics is necessary in the classical domain but that one can 'do' quantum statistics without individuals at all. In the latter case, in particular, some innovations regarding individuality and the nature of objects will be required.

[185] Hence in this respect we disagree with Huggett (ibid., pp. 7–8) although we are obviously sympathetic to his overall theme concerning the problematic matter of drawing metaphysical conclusions from physics.

We see then that there are a number of alternative solutions to the problem of how to assign the required classical weights to the states represented by a point in occupation number space. If the axiomatic approach is adopted then it is simply assumed that each such point has a factor $n!$ assigned to it and the appropriate weights for particle permutations are assigned on the basis of some form of non-spatio-temporal TI. If, however, one feels compelled to offer some further justification of these axioms, then one can effect a transformation to the $r$-dimensional phase space where this point is exploded out, giving the weighting factor without having to resort to this form of TI. The permuted complexion is now regarded as distinct from the unpermuted one on the basis of the system's continuous trajectory in this phase space and individuality is conferred via a form of Space-Time Individuality. Of course, in this latter case, one must still justify the assignment of equal probabilities to the points of this phase space through some form of ergodic hypothesis. Finally, one could stay with occupation number space, resist the transformation to phase space and obtain the appropriate statistics via Huggett's strategy above. Insofar as this depends on an assumption concerning the occupation numbers which is supported either by the Impenetrability Assumption or considerations of the spatio-temporal background, individuality is grounded on STI here too.

Returning to the language of particles, on this account there is no underlying substance conferring individuality or guaranteeing impenetrability. The last is merely one of a set of properties predicated of the particle, moving along a well-defined continuous trajectory in space-time. Of course, some of these properties may change; in particular, 'extrinsic' properties are dependent on the state of the particle.[186] Intrinsic, state-independent properties, such as rest-mass, charge etc., do not change and allow us to re-identify the particle as a particle of the same *kind*.[187] With regard to reidentifiability, it is these properties which satisfy the sortal constraint by defining the relevant sortal term, such as 'electron' for example. Together with continuity, what we have on this account is nothing more than a set of properties and an underlying sequence of space-time points at which this set is instantiated. Of course, by grounding the individuality of the particle in the points of space-time in this way, this view pushes us to consider the nature of these points themselves and, in particular, *their* individuality.

[186] Jauch 1966, ch. 15.

[187] Jauch ibid.; see also Hirsch 1982, p. 31.

## 2.4 INDIVIDUALITY AND SPACE-TIME

There is, of course, an immense literature on the nature of space-time and we have no intention of trying to do justice to it here. We shall simply take what we need from this body of work in order to illustrate the relevant issues to do with identity and individuality in this context.

First of all, if a relationist account is adopted in which spatio-temporal relations are regarded as ultimately reducible to non-spatio-temporal relations between objects, then STI may not even get off the ground in the face of the obvious circularity involved. On such an account the Principle of Individuality for objects must be sought elsewhere.[188]

Equally clearly, STI is compatible with the alternative substantivalist view according to which space-time is a form of substratum itself with features which are independent of the existence of material objects.[189] Of course, one can still hold a non-spatio-temporal form of TI in this context, with spatio-temporal continuity allowing us to *distinguish* the particles and hence infer their individuality. If one wants to adopt STI, however, then one should say something about the underlying space-time points themselves. Should these also be regarded as individuals and if so, what confers their individuality?

There is an obvious difficulty in answering these questions. In the case of physical objects, whatever account of individuality we adopt, we begin with distinguishability and then either make the move from epistemology to ontology, or equate the two. In the case of space-time points, there is an issue as to whether they can even be regarded as *distinguishable*.

In both classical physics and special relativity, where the space-times are flat, there is nothing to distinguish one space-time point from another. As Anderson has famously put it:

> The distinguishing feature of a particular point of … space-time is that it has no distinguishing features; all points of space-time are assumed to be equivalent.[190]

Of course, one could still maintain that the space-time points are nevertheless individuals, with their individuality conferred by the underlying,

[188] Leibniz famously understood it to be embodied in his Complete Notion of an Individual, from which PII follows.

[189] It is worth noting that arguments for the substantivalist view of space-time bear a close resemblance to those for substance in general (Sklar 1977, ch. 3; Sklar op. cit., p. 161).

[190] Anderson 1967, p. 4.

non-material substratum but there is an obvious irony in grounding STI on such a claim. In the context of General Relativity, on the other hand, there is the possibility of the points being distinguished through the curvature of space-time. This suggestion has been raised in the context of the debate whether curvature is intrinsic to space-time or not,[191] which in turn can be situated in the broader debate concerning what we take 'space-time' itself to be. Pursuing the details of these debates would take us too far from our central theme but we feel it is worth illustrating the role that considerations of identity and individuality have played in these discussions, through the following exchange between Grünbaum and Stein.

They address, in particular, the question, 'whence does the space-time manifold get its particular metric structure?'.[192] Typically, but not necessarily, substantivalists will argue that this structure is intrinsic or, in some sense, 'internal', to space-time, whereas relationists will respond that it is imposed from without, by appropriate metric standards suitably embodied. Stein, however, raises the issue as to whether we can even appropriately *distinguish* this metric structure, as somehow separate from space-time, to begin with; that is, he is concerned that the question regarding the ontological status of the metric may beg too many questions both with regard to its relationship with the space-time, however construed, and with regard to our (theoretical) access to space-time.[193] There is a contrast here, Stein claims—and this is where considerations of distinguishability and individuality emerge—between everyday objects, such as a globe sitting on a desk, and space or space-time. In the former case, we can sensibly ask whether the globe possesses certain *intrinsic* metrical properties[194] because we have access to it, independently of these properties. The problem is, he insists, we have no such independent access to space-time.

Thus Stein writes,

> … if we ask (assuming Newtonian physics) whether 'equality of time-intervals' is a relation *intrinsic to the space-time manifold*, and if this is construed (roughly) to mean 'whether that relation is involved in the structure of *the space-time manifold itself,*

[191] See Salmon 1977; Nerlich 1979.

[192] Grünbaum 1977, p. 332.

[193] Ibid., pp. 327–8. The structure of the dialectic here is interesting: Grünbaum presented a version of his paper at a conference on 'Absolute and Relational Theories of Space and Space-Time' held at the Boston Institute of Relativity Studies, 3–5 June 1974. The published version contains oral responses from Stein to questions put by Grünbaum at the conference. Stein's own paper in the same volume consists of an extract from a letter to Grünbaum in response to the latter's request for comments on his paper and incorporates further remarks occasioned by the subsequent revision of Grünbaum's paper!

[194] Such as that of having a surface which is a Riemannian 2-manifold of constant positive curvature (Grünbaum 1977, p. 328).

> *considered apart from all other entities*', the question at once arises of how to explicate the notion of 'the space-time manifold itself', and of the conceptual line between it and 'all other entities'. I see no way to confront the former question independently of the latter; and yet the converse may also seem to hold: that we cannot give a conceptual explication of 'the space-time manifold' without begging the question of its intrinsic properties.[195]

In response, Grünbaum appeals to the example of a man named 'Jack' who happens to be an uncle, in order to draw the well-known distinction between intrinsic and extrinsic properties.[196] Thus, Jack possesses the property of being an uncle only in virtue of there being someone else, 'external' to him, of whom he is their uncle. 'Unclehood' is ontologically dependent on a relation to an object external to Jack, whereas the property of bipedality is not. In other words, unclehood is extrinsic, whereas bipedality is intrinsic. Now, continues Grünbaum,

> … even after Jack has been explicitly characterized as an uncle upon having been *identified* as "Uncle Jack", such identification does *not at all beg the question* whether his unclehood is ontologically intrinsic to him in the manner of his genuinely monadic property of bipedality![197]

Putting this into the scholastic terms we have employed here, the use of certain properties in *distinguishing* an object should not then beg any questions as to the ontological status of such properties.

This becomes even clearer in the further example Grünbaum gives of distinguishing two individuals: Presidents Ford and Giscard d'Estaing (remember them?!). Each can be characterized and identified in terms of certain properties—President of the USA and President of France respectively—and, Grünbaum insists, regardless of which particular properties effect these identifications, one can determine in a non-question-begging manner which properties are intrinsic and which are not. What is important for distinguishing the two is not whether the properties involved are intrinsic or extrinsic, but that there is *some* difference between their properties:

> *By invoking a particular set of properties initially in a definite description to identify these (human) objects, one does not thereby prejudge or determine the degree (monadicity vs.*

[195] Quoted in Grünbaum ibid., p. 329.

[196] Ibid., pp. 332–3.

[197] Ibid., p. 333.

*polyadicity) which these and the properties of theirs will turn out to have subsequent to the identification.*[198]

This analysis is then carried over to the case of space-time and Grünbaum insists that the fact that certain metrical properties feature in the identifying characterization of space-time does not prejudge the question whether such properties are intrinsic, by analogy to Jack's bipedality, or extrinsic and ontologically relational, by analogy with his unclehood. Granted that we may have *easier* epistemic access to globes on desks, say, than to the metrical feature of space-time, this difference

> … cannot serve to sustain Stein's charge that owing to the unavailability of an independent ostensive definition, any theoretical identification of physical space-time ineluctably begs the question concerning the ontological intrinsicality (monadicity, absoluteness) of its identifying properties.[199]

We shall come back to Stein's response below, since both this response and the above debate in general offer illuminating comparisons with the situation regarding particles. For the moment, let us return to the question of the individuality of space-time points. In the case of particles our metaphysical considerations were grounded in a feature of scientific practice, namely the counting as distinct of arrangements formed by permuting particles over states. Is there anything analogous in the physics of space-time? Grünbaum has argued that there is and that it can be found in the physicists' generation of so-called 'covering spaces' by the 'disidentification' or explosion of points in a given manifold.[200]

Consider again an example of an 'ordinary' object: a hemisphere sitting on a table in Euclidean space.[201] This can be turned into a model of elliptic 2-space by identifying the antipodal equatorial points of the surface. Nevertheless, in this case, Grünbaum insists that it is both *meaningful* to assert that the antipodal points are in fact distinct, since they coincide with different points of the table, for example, and that, furthermore, we are *epistemologically* able to determine that they are distinct. The suggestion, then, is that both of these claims carry over to the physics of space-time.

Thus, as Grünbaum notes, Schrödinger considered the warrant for making a similar identification of the antipodal points of a pseudo-sphere in De Sitter

[198] Grünbaum 1977, p. 335; emphasis in original.

[199] Ibid., p. 337.

[200] Hawking and Ellis 1973, p. 181.

[201] Ibid., p. 365.

space-time to give an elliptic interpretation. Moving the other way, one can 'disidentify' the points of a manifold to give a *topologically different* covering manifold. Hawking and Ellis, for example, appeal to just such a procedure in order to obtain the property of time orientability in the case in which a particular space-time lacks it and take it as an assumption that either the space-time is time-orientable or one can deal with the time-orientable covering space.[202] How should we regard these different models? Glymour argues that they must be understood as contradictory and irreconcilable on the grounds that the alternative topologies depend on the basic individuals of the models and that such differences are matters of truth or falsity.[203] Hence it is meaningful to ask whether the space-time points of 'our' universe are distinct, in the same sense as the antipodal points above are distinct, or not and consequently, meaningful to ask which of the topological alternatives we find ourselves in. Having established this much, it is then a further epistemological question whether we can ever discover which space-time we are in.[204] Both Glymour and Malament have argued that for certain space-time models, there can be no empirical evidence which could ever resolve this issue.[205]

The crucial question then, as Grünbaum asks, is: "what criteria of identity or distinctness for [space-time points], if any, can give *physical meaning* to the required formal disidentifications at the ontological level of postulated space-time theory?".[206] In particular, an adequate defence of the claim that the above questions are at least meaningful "depends on the provision of a viable criterion of individuation for [space-time points] which are *prima facie* so much alike with respect to their monadic properties".[207]

So, what criteria of individuation might there be? Let us consider the Principle of Identity of Indiscernibles first of all. As Stein has noted, Newton also accepted this principle and took it to apply to space and time.[208] On this basis, in order to be considered as individuals, the points of space, time and space-time must possess some distinguishing property or relation. For

[202] op. cit.; Grünbaum op. cit., p. 366.

[203] Glymour 1977.

[204] Grünbaum op. cit., pp. 365–6.

[205] This raises obvious problems for the realist (Torretti, R., personal communication 1986). The anti-realist could either take the positivist route and simply deny that it is meaningful to ask which model we are 'in', or, more sophisticatedly perhaps, take the path of the constructive empiricist and agree that it is indeed meaningful to ask such a question, since there is a truth of the matter, but deny that we could ever know that truth.

[206] op. cit., p. 366.

[207] Ibid.

[208] Stein 1967, p. 184; Grünbaum op. cit., p. 364.

Newton, the 'parts' of space could be regarded as distinguished and, hence, individuated by their 'internal' relations:[209]

> … just as the parts of duration derive their individuality from their order, so that (for example) if yesterday could change places with today and become the latter of the two, it would lose its individuality and would no longer be yesterday, but today; so the parts of space derive their character from their positions, so that if any two could change their positions, they would change their character at the same time and each would be converted numerically into the other. The parts of duration and space are understood to be the same as they really are because of their mutual order and position; nor do they have any hint of individuality apart from that order and position which consequently cannot be altered.[210]

But, of course, it is the apparent unavailability of any independent notion of the 'position' of a point of space that led Leibniz to deploy the Identity of Indiscernibles *against* the Newtonian view, arguing that since such points are considered to be *indistinguishable* in terms of their intrinsic, monadic properties, they must all be *identical*, in the strict sense. As this is absurd, the notion of space as composed of such points must be rejected in favour of a relationist view in which 'external' relations involving non-spatial bodies serve to individuate the points of our model, regarded as a mathematico-physical description only.

What about the situation in General Relativity? Can the curvature of relativistic space-time provide sufficient heterogeneity to allow for the points to be distinguished, and hence individuated via PII? Grünbaum considers a method of constructing an intrinsic coordinate system using the metric which, if it could be applied globally, might be up to the job.[211] Unfortunately, he acknowledges, it can't and it isn't, since this method works only locally and even then

[209] There is an interesting question regarding the distinction between 'parts' and 'points' in this context: did Newton have the modern concept of a point as dimensionless? Koslow has argued that he did not and that throughout Newton's writings he refers to 'minimalia' or 'indivisibles' which were taken to be small parts of least magnitude. If space is composed of such minimalia, then it can obviously be understood as possessing an intrinsic metric; hence it comes as no surprise to find Grünbaum opposing this historical interpretation (op. cit., pp. 305–13). There is an obvious comparison to be made with Leibniz who famously struggled with, or through, the 'labyrinth of the continuum'. Arthur has attempted to provide an understanding of Leibniz's apparently contradictory remarks by suggesting that he rejected the view that space is *composed* of points in favour of a notion of space being partitioned into cells (called 'natural machines' by Leibniz), where such partitioning can be continued without limit and points are regarded as line boundaries and not the end results of the limit process. It is further claimed that sense can be made of this notion by recasting it in the formal framework of combinatorial topology (Arthur 1986).

[210] Newton in Hall and Hall op. cit., p. 136; cf. Newton 1934, p. 8.

[211] op. cit., pp. 366–7.

it may not be able to individuate. Here, then, we face the problem noted above: we simply don't have a grip on the distinguishability of the space-time points in the first place and therefore don't have the basis for appealing to PII in order to guarantee their individuality.

Perhaps, then, we should abandon the search for some distinguishing features as hopeless[212] and directly assert that the space-time points are individuals, understood in terms of primitive thisness, for example. Grünbaum himself rejects such a proposal, on the time-honoured grounds that the property of self-identity satisfies PII only trivially and cannot therefore be accepted as an individuating property.[213] As we have noted, however, primitive thisness can be effectively disconnected from PII and invoked as a principle of individuality in its own right, as it were. It is interesting, therefore, that Hoefer has recently identified just this notion of primitive thisness as laying at the core of a further, much discussed issue in space-time physics which, similarly to the covering space example above, also involves the generation of alternative space-time models which cannot be distinguished empirically.[214] This is the issue concerning the implications of the famous 'hole argument'.[215]

Given a space-time model in General Relativity, another model can be generated which is identical to the first for all points outside a certain region—the hole—but not inside, through the application of an appropriate diffeomorphism to the points of the underlying manifold. Now, if substantivalism is understood to involve the identification of space-time with the underlying manifold—on the grounds that in General Relativity geometric structures, such as the metric tensor, are physical fields 'in' space-time[216]—then the substantivalist must accept these two qualitatively identical models as physically distinct. And this distinctness goes beyond observation or what can be determined on theoretical grounds. Hence substantivalism is ontologically profligate and, the argument goes, should be abandoned.[217]

[212] It may not be entirely. Grünbaum notes that Glymour has suggested that for certain space-times in which the two lobes of the light cone are disjoint, metric relations may serve to distinguish and hence individuate (Glymour 1972; Grünbaum op. cit., p. 367). However, it is unclear whether this criterion of individuality can serve to distinguish between a space-time and its covering model and thus answer Grünbaum's question as posed above.

[213] op. cit., p. 364.

[214] Hoefer 1996.

[215] The argument can be traced back to Einstein but in its current incarnation was spelled out in a well-known paper by Earman and Norton 1987.

[216] Earman and Norton 1987, p. 518.

[217] If the situation regarding the hole is chosen appropriately, it turns out that the two models can agree up to a certain time but then diverge afterwards, giving what is regarded as a failure of determinism. This has been taken as a form of reductio of substantivalism on the grounds that if determinism fails it should do so for a reason to do with physics, not metaphysics. See, e.g., Earman and Norton op. cit.; Earman 1989, p. 180; Butterfield 1989; Maudlin 1988. We shall be less concerned with this conclusion than with the implications of the argument as a whole for our discussion of the individuality of space-time points. Hoefer, for example, has questioned the nature of and grounds for the above implication; op. cit.

It is worth noting, as Hoefer does,[218] that a diffeomorphism can be seen as a *permutation* of the points of the manifold which satisfies certain restrictions. This provides an obvious point of comparison with the situation in particle physics. In the hole argument a permutation of the space-time points leads to a new model, analogous to the new arrangement generated by a permutation in classical statistical mechanics. In the latter case, this new arrangement is counted as distinct and if it were not, the statistics would be very different. In the case of space-time physics, how are models generated by such diffeomorphism regarded? It turns out that physicists do not regard such models as physically distinct, since they are qualitatively indistinguishable, the only difference concerning what fields are located at what space-time points. This attitude has been interpreted as an endorsement of 'Leibniz Equivalence', which holds that two such models are equivalent in the sense of representing the same physical situation.[219] Substantivalism, of course, is committed to the denial of Leibniz Equivalence, just as the Newtonian absolutist was earlier committed to the denial of the claim that a description of the universe as it is, is equivalent to a description of the universe with its material context moved 10 feet in a given direction in space.

If such a space-time model is regarded as describing a possible world, then denying Leibniz Equivalence in the context of the hole argument implies the acceptance of haecceitism.[220] In the present context, this holds that two space-time models may not differ qualitatively in any way, yet still differ in what they represent *de re* concerning the individual space-time points. Hoefer goes further in claiming that since acceptance of haecceitism entails acceptance of primitive identity, substantivalists, in denying Leibniz Equivalence, ascribe primitive identity to space-time points:

> … two such models can only represent different physically possible worlds if we believe that space-time points (or regions) not only exist, but have primitive identity, and so could have all of their properties systematically exchanged with the properties of other actual points …[221]

[218] op. cit., pp. 7 and 8.

[219] Earman and Norton op. cit.

[220] Hoefer op. cit., p. 15; recall our discussion above and the outline in Chapter 1.

[221] Ibid. Here Hoefer appeals to the example of two dice, concerning which the question is asked: does it make sense to ask whether dice A could have all the properties of dice B and vice versa? (He may have obtained this example from Kripke, who uses it in his discussion of rigid designation; we shall return to this later.) As he notes, anyone who holds that the dice are individuals, and in particular if such individuality is understood in terms of primitive identity, will answer 'yes' to this question, although the advocate of STI will of course insist that care be taken in spelling out which properties are included.

This entailment is problematic. According to Lewis, as we have already noted, a belief in haecceities is neither necessary nor sufficient for haecceitism. If haecceities are understood as Adams understands them, namely as primitive thisnesses involving primitive identity, and Hoefer explicitly draws his notion of the latter from Adams, then one might assert haecceitism but deny primitive identity, on the grounds that Lewis has laid out, which involve a form of nominalism. In that case, of course, our substantivalist is going to have to appeal to some other Principle of Individuality consistent with these nominalist grounds; one such might be 'bare' substance. This gives a rather curious account of the individuality of space-time points but it is one that the substantivalist could cling to if necessary. The STI alternative is obviously not an option!

Alternatively, one could try to resurrect the idea that this individuality is grounded on the properties—that is, the *structural* properties—of the space-time points themselves. Maudlin has offered a version of this idea which is a form of metric essentialism: space-time points bear their metrical relations essentially.[222] This undermines the basis of the hole argument since the diffeomorphism is not now taken to generate a possible situation in the first place.[223] Concomitant with this idea is a shift in what we take space-time to be: for Maudlin it is not simply the manifold but the latter *plus* the metric structure. Unfortunately, however, Norton has argued that if space-time is understood as the manifold plus further structure and if this further structure allows of certain common symmetries, then a version of the hole argument can be revived.[224] In particular, such symmetries arise in Newtonian space-time theory and in Special Relativity and General Relativity as applied to spatially homogeneous and isotropic cosmologies. As Norton says, the crucial point

[222] Maudlin 1989, p. 86. Maudlin claims ancestry for this view in the passage by Newton quoted above on p. 45; see also Earman 1989, pp. 218–19 for further discussion. A nice account of the way in which Newton's and Leibniz's views have been inaccurately portrayed and framed in terms of the substantivalist–relationist debate in given in Disalle 1994.

[223] We have already noted how the issue of what counts as a possible situation arises in these discussions.

[224] Norton 1989. Norton's argument crucially hinges on the claim that, as in the case of pure 'manifold substantivalism', incorporating further structure does not prevent the substantivalist from being committed to the distinctness of observationally indistinguishable states of affairs (op. cit., p. 60).

here is that

> … the presence of these symmetries represents a failure of the further structure to individuate fully the points of the manifold.[225]

There are other responses to the hole argument, of course. Butterfield also focuses on the relationship between space-time models and possible worlds and argues that at most one of the two models related by a diffeomorphism can represent a possible world.[226] He justifies such a move by denying the trans-world identity of space-time points and argues that this allows him to revive a form of determinism compatible with substantivalism. Norton gives two responses to this approach:[227] first, Butterfield's argument conflicts with (Norton's understanding of) scientific practice, which takes *any* structure of the appropriate type satisfying the field equations as a model of the theory.[228] In particular, if $M$ is a model of General Relativity and $M'$ is obtained by applying a diffeomorphism, then $M'$ is also a model and there are simply no grounds for choosing the former as representing a physical situation and not the latter. Secondly, Norton raises the issue of underdetermination and asks how are we to distinguish $M$ and $M'$? He writes,

> [t]here must be some property which distinguishes them and the property must be physically significant in so far as it tells us which structure represents a physically possible world.[229]

But since $M$ and $M'$ are diffeomorphic, this property cannot be cashed out in observational terms; nor can it be cashed out in terms of the relevant laws, since the field equations themselves do not distinguish between the development into the hole of $M$ or of $M'$.

Butterfield's counter-responses are interesting for what they reveal about the fundamental intuitions in play regarding possibility. With regard to Norton's first point he insists that not all the models of space-time theory represent possible worlds.[230] Which models represent worlds is not given by scientific practice but by the combination of epistemology and metaphysics in terms of which we interpret the theory. His answer to the second response essentially recapitulates the discussion of what counts as a genuine possibility. In particular, Butterfield argues, only one geometric object is needed to code essential

[225] Ibid., p. 60. Maudlin's response is that although such symmetries do indeed arise for certain models of General Relativity, in our world they are unlikely; Maudlin 1993.

[226] Butterfield 1989.

[227] op. cit., pp. 62–3.

[228] Ibid., p. 62.

[229] Ibid., p. 63.

[230] Norton 1989, p. 79.

properties of points in order for us to be able to infer that at most one model represents.

These answers are problematic. Physicists do appear to be happy to use either one of two observationally indistinguishable models,[231] as the covering space example suggests. Furthermore, Butterfield's response here falls prey to the standard anti-realist critique: any factor selected by the realist or substantivalist for choosing one model over another, that is not grounded in the *evidence*, will be dismissed by the anti-realist or anti-substantivalist as 'merely' pragmatic. Concerning his second counter-response, arguments as to what counts as 'genuine' possibilities are always liable to beg the question against the opposition. What is needed in this case is for Butterfield's encoding object to be displayed.[232]

Hoefer's response is more radical than either Maudlin's or Butterfield's: he denies that the space-time points have primitive identity.[233] Of course, given what we've just said, this doesn't in fact resolve the issue, since one might adopt an alternative account of the individuality of space-time points; what is required, it seems, is the denial of *haecceitism* in general. We shall return to this point shortly.

As Hoefer recognizes, this denial of primitive identity faces a number of challenges, not least of which is one bound up with our remark that an account of individuality of space-time points without primitive identity would be curious. And its curious nature is in fact brought out by the comparison Hoefer makes with the case of particles in an effort to respond to the challenge that primitive identity is surely part of what it means for something to be a substance.[234] He first of all appeals to our now familiar balls and boxes example and insists that we do not need primitive identity to make sense of questions such as 'which ball entered which box?' since we have recourse to the space-time trajectories of the balls. He then generalizes to the example of unbiased dice and insists that the basis for the assignment of the appropriate statistics is not an assumption that the dice have primitive identity, but rather the 'fact' that they have distinct continuous trajectories. Space-time points, of course, do not have such trajectories. If they did, he claims, then they might be capable

[231] The phrase is Malament's and should be understood in the sense that inhabitants of the associated space-times cannot tell by observation which space-time they are inhabiting, and not, of course, that the models themselves are the objects of observation.

[232] Further discussion of responses to the hole argument can be found in Earman op. cit., ch. 9.

[233] Ibid., p. 15. As he acknowledges, such a suggestion has been previously made by Maidens (Maidens 1993).

[234] Ibid., pp. 15–18.

of exhibiting Maxwell-Boltzmann statistics.[235] but this would not mean that they have primitive identity. He concludes:

> What goes for points, goes for particles. Even if atoms had distinct and continuous trajectories, we would not have to ascribe primitive identity to them in order to think of them as real substances. The ascription of primitive identity allows us to pose certain strange philosophical questions—but not to do any more productive work.[236]

This is an odd conclusion. One would think it would go the other way: what goes for particles, goes for points. As we shall see, arguments have been given for a similar 'loss' of individuality in particles, based on the counting inherent in quantum statistics where different arrangements obtained by particle permutations are not counted as distinct (it should be noted that such arguments and their further discussion precede Hoefer's, of course).[237] In the absence of a similar sort of argument here, Hoefer's response seems somewhat *ad hoc*. He is right, of course, that we can always ground Maxwell-Boltzmann statistics, as applied to balls in boxes, dice and classical atoms, in Space-Time Individuality but, as we have also noted above, adherents of non-spatio-temporal versions of TI will insist that the 'fact' that such entities possess well-defined trajectories pertains to their distinguishability only; their individuality rests on something else. The lack of clarity between distinguishability and individuality is apparent from the last sentence in the above quote; of course primitive identity does no 'productive work' when it comes to distinguishability, but, its defenders would insist, it does *all* the metaphysical work in grounding individuality.

This last point is important because Hoefer adheres to the view that individuality needs no such grounding, that it can be taken as primitive. This emerges in his response to the further challenge: can we make sense of a notion of substance *sans* primitive identity? (Again, as we shall see, this is something we shall come back to in the particle case when we come to quantum physics.) Not surprisingly, Hoefer thinks we can. Thus he notes that the claim that two objects are individuals may imply that they can be distinguished in terms of their properties. However, he insists, it may not: in the case of the possible world with twin globes which is typically presented as a counter-example to PII, it makes sense to speak of the globes as individuals, according to Hoefer,

[235] Earman, op cit., p. 17.

[236] Ibid., pp. 17–18.

[237] Indeed, such arguments on the particle side motivated Maiden's rejection of individuality on the space-time side.

because we have stipulated that there are two of them and not one. Nothing more is needed:

> To suppose that there is something more to the claim that *A* and *B* are individuals is, I submit, merely to insist on primitive identity for no clear reason. I believe we can do without it.[238]

However, first of all, to suppose that there is something more here is not to insist on primitive identity, as we have noted already. And secondly, it is not to *merely* insist on primitive identity, or any other principle of individuality, 'for no clear reason'! Without such a supposition, the stipulation that there are two globes is entirely question begging when it comes to the status of PII. As Hacking has insisted, in these sorts of cases, bare stipulation is not enough, what is needed is argument.[239] More generally, the claim that individuality is primitive and requires no further analysis obscures the relationship between individuality and distinguishability. One may, of course, adopt as primitive whatever term or notion of one's framework one likes; what is important is how useful or illuminating such an adoption is. Moving back one further step, to the metaphysical level of TI and STI, helps to illuminate the distinction between individuality and distinguishability and opens the door to further interesting metaphysical possibilities, as we shall see.

But if we grant that we can make sense of this notion of primitive individuality without primitive identity, how are we to interpret the manifold of our space-time model? Mathematically, this manifold is a set of points possessing a certain topological structure. If we accept that the theory of General Relativity quantifies over these points, then it would seem that we are committed to their existence as distinct individuals.[240] There are two options one could choose between at this point. One could agree that the theory quantifies over the points, but insist that they do not possess primitive identity. Since the latter involves, or rather, is nothing but, self-identity, this position requires not only a new understanding of quantification but also a non-classical logic and set theory—that is, formalisms which can accommodate the notion of non-self-identical points. We shall return to such a possibility in the discussion of quantum particles where such formalisms have been developed (prior to and independently of these space-time considerations).

[238] Ibid., p. 19. In a footnote here, Hoefer suggests that his argument is essentially the same as that given by van Fraassen, for example, against the view that quantum particles have 'lost' individuality. As we shall see in later chapters, this suggestion is misguided.

[239] Hacking 1975.

[240] Hoefer op. cit., p. 23.

The alternative is to adopt a more fine-grained approach to what the theory quantifies over and acknowledge that one could still give a realist construal of the theory in which it is interpreted as supporting the existence of space-time, but that the latter should not be identified with the manifold (we recall that in the context of the hole argument this is precisely the identification made by substantivalists).[241] It is time (past time!) to return to the Grünbaum–Stein debate, in which the issue as to what aspect of our model we take to be 'space-time' features so prominently. We recall Stein's concern that the question as to whether the metric is intrinsic or not might be a pseudo-question, since we have epistemic access to space-time only via this metric. As we saw, Grünbaum responded to this concern with his 'Uncle Jack' and two Presidents examples to indicate that we could have access to, and distinguish objects, via certain sets of properties, whilst maintaining that certain of these were intrinsic to the object. In his counter-response, Stein clarifies his concern and reveals a fundamental difference between himself and Grünbaum with regard to the ontology of space-time.

Thus with regard to the Uncle Jack and globe examples he notes that we are only able to get the process of conceptualization in terms of intrinsic and extrinsic properties off the ground in the first place because we are already familiar with these examples, qua *objects*. They belong to a kind "… about which I possess an abundance of lore (or prejudice), namely *bodies* and *people* (antirespectively); and are quite unlike space-time, about which I happen to be deficient of such prejudice".[242] As we have already said, we can distinguish, or pick out, Uncle Jack or a globe on the basis of their properties and then go on to discuss which of these properties are intrinsic or not but it is not clear whether we can even complete the first stage and unequivocally delineate or distinguish *space-time* to begin with. As Stein points out, different delineations of space-time will give different answers to the question whether the metric is intrinsic. Thus, for example, someone might distinguish space-time as 'the mere four-dimensional differentiable manifold, independently of any further structure' (recall Hoefer's characterization of the substantivalist view above), whereas someone else might distinguish it as 'the smooth 4-manifold with

[241] It is not entirely clear which of these two alternatives Hoefer prefers. At one point he suggests that a substantivalist could deny primitive identity but not the existence of space-time points. However, he gives no indication as to how this could be formally accommodated. Moreover, he explicitly rejects the quantification over these points and the final sections of his paper seem to indicate a preference for the alternative.

[242] op. cit., p. 394. In the case of particles, statistical mechanics substitutes for this 'lore'.

the distinction of directions at each point into *space-like* and *causal* (time-like or null) directions'. For both such persons, the metric is extrinsic but for the second the conformal structure is intrinsic, and Stein insists that not only does he not possess any criterion for deciding between such characterizations, he sees no point in seeking one.[243]

This is the nub of the matter. Stein understands Grünbaum as thinking of the world in terms of things or 'primary substances', whereas he thinks in terms of *structures* or *aspects of structure*. Space, or more generally, space-time, is "… *an aspect of the structure of the world* ".[244] On this view, with space-time not even conceptualized as a distinguishable thing to begin with, the question as to what is intrinsic amounts to no more than a request for which aspects of physical structure we denote as spatial. In particular, Stein insists that we must not let our ontology be driven by our mathematical representation.[245] That we separate the manifold from the metric and stress-energy tensors in our model of space-time does not justify any implication that the manifold somehow 'carries' or ontologically underpins the metric field. Of course, the field has to be defined 'on' the manifold to make mathematical sense but this should not be taken as carrying any extra metaphysical baggage.[246]

Similarly, Hoefer's second alternative above can be better understood as structuralist in this sense. Having rejected the characterization of space-time in terms of the manifold alone, with the attendant problems concerning the individuality of the associated points, he advocates a view of 'metric field

[243] Hoefer, p. 395.

[244] Ibid., p. 397; his emphasis. Stein offers this as an interpretation of a well-known passage in one of Newton's unpublished papers where he insists that space "… has its own way of existing, which fits neither substances nor accidents" (p. 131 in Hall and Hall 1962—Stein notes that the Halls' translation is marred by a certain lack of philosophical perspicuity; op. cit., fns 12 and 13, p. 402). It is not a substance because it does not subsist 'absolutely of itself' and also does not have the characteristics of a substance. It is not an 'accident' because space does not 'inhere' in some subject, as accidents do. It is not nothing at all, because nothing can have no properties but we can conceive of space having properties (and in that sense it is close to substance). Thus, Newton concludes, 'Space is an affection of a thing *qua* thing' (1962, p. 136; Stein op. cit., p. 396) and Stein understands this as the doctrine that "… *the fundamental constitution of the world*—its 'basic lawful structure'—*involves the structure of space*, as something to which whatever may exist must have its appropriate relation" (ibid., p. 396; his emphasis). Space, then, is 'an aspect of the nature of thinghood' (ibid.) and placed in a modern setting, what this means is that 'Whatever exists of a physical nature … must be appropriately related to a space-time manifold with a fundamental tensor-field satisfying the Einstein equations' (ibid., p. 397). Furthermore, Stein argues, this structuralist view is also expressed by Riemann in his talk of the 'real' or 'actual' that underlies space (ibid.).

[245] Cf. Maudlin (op. cit., pp. 82–3) who similarly insists that we need to distinguish the representation from the object represented.

[246] Ibid., p. 399.

substantivalism' in which the 'real representor' of space-time is the metric field:

> The division of the representors of the properties of space-time we see in describing space-time models as triples (*M, g, T*) does not indicate any deep dualism in the nature of space-time itself, on the view that I am describing here. Space-time has physical continuity, topology and metrical structure.[247]

The focus on the metric field is justified in two ways: first there is an asymmetry between the metric and the manifold in that to give the former without the topology associated with the latter is still to describe part (albeit only the local part) of space-time, whereas to specify the manifold without the metric is not to give the space-time at all. Secondly, all the empirically useful and explanatory work within the theory of General Relativity is primarily done by the metric field. It is this which leads the substantivalist to claim that space-time cannot be given a relationist construal in the first place.[248]

This offers a way around the hole argument by denying the initial premise that space-time be identified with a manifold of points whose permutation has physical significance. However, the structural characterization brings out the issue that it is not so much primitive identity that needs to be denied as *haecceitism* in general. Of course, this move entails the acceptance of Leibniz Equivalence and we may paraphrase Stein here and insist that we see no need to choose between the models. This is *not* to reduce this view to relationism if the latter is understood in the standard form in which the manifold is constructed from physical events;[249] rather it is the relevant relations themselves that are accorded separate ontological status.[250] It may also offer a response to Grünbaum's request for a criterion of identity or distinctness for space-time points which can give physical meaning to the disidentifications involved in generating cover spaces. No such criterion is to be sought for because such disidentifications are not regarded as having any *physical* meaning in the first place, only a mathematical one. What is important, and what one should be a realist about on this view, are the relevant *structures* involved.[251]

[247] op. cit., p. 24.

[248] Cf. Maudlin op. cit., p. 87.

[249] See Earman op. cit., pp. 194–5. The concluding remarks of Disalle's historically informed analysis are pertinent here (op. cit., pp. 278–87).

[250] It is a further interesting question whether this position differs significantly from Earman's attempt to do away with space-time points in terms of 'Leibniz algebras' (op. cit., pp. 191–4).

[251] Cf. Ladyman (1998).

Of course, there is a great deal more which can be said here[252] but we have strayed far enough from our concern with STI. The point of this discussion was, first, to indicate the issues which arise if the individuality of particles is taken to be grounded in space-time and secondly, to emphasize that these issues have been addressed by reflecting on the analogies and disanalogies with particle individuality. If circularity is to be avoided, some form of substantivalism appears to be the appropriate view to adopt. But that raises profound questions to do with the individuality of the points of the underlying manifold. One option is to keep the points but stop short of regarding them as individuals, a move which requires an alternative formal framework. Another option is to side-step the individuality issue by effectively recasting our characterization of space-time in terms of structures, rather than objects or things. We shall encounter these options again in the discussion of quantum particles, to which we shall shortly turn.

## 2.5 CONCLUSION: THE METAPHYSICS OF CLASSICAL PHYSICS

Let us briefly return to the argument with which we began this chapter: classical particles should be regarded as individuals because of the way permutations are counted in classical statistical mechanics. As we indicated, this argument has been resisted on the grounds that, for example, the Gibbs Paradox indicates that this counting is deeply problematic. These grounds can be spelled out as follows: we begin with the requirement that the entropy of a system be 'extensive'. This means that if the volume of some system (a gas, say) is divided into two halves, then the entropy of the whole system must equal the sum of the entropies of the respective halves. Unfortunately, the expression for the entropy obtained from Maxwell-Boltzmann statistics does not in fact meet this requirement. Thus when it is actually applied to the situation where a sample of gas is mixed with a sample of the same kind of gas, it yields an entropy increase. This is unsatisfactory on two grounds. First, and foremost, it just does not agree with the results of classical thermodynamics. Secondly, and more dubiously, it might be argued that in so far as classical thermodynamics

[252] In so far as the covering space may involve a different topology, the structuralist is going to have to regard such differences as irrelevant from a realist point of view. Whether she can do so and still lay claim to being a realist is a moot point.

is a 'phenomenological' theory in which no reference to particles need be made (and of course such an argument might well draw on the history of the subject at this point), nothing 'physical' can be said to happen when two samples of the same gas are mixed. Thermodynamically speaking, 'mixing' is a concept which is only applicable when different kinds of gases are brought together and so from this perspective, the mixing of a gas with itself is not a physical event; hence the entropy change should be zero.

Standardly, the 'paradox' is resolved by excluding the relevant permutations from the calculation of the statistical entropy. This can be achieved by simply dividing the classical expression for the number of possible complexions by N!, yielding an expression for the entropy—known as the Sackur-Tetrode equation—which is extensive as required.[253] The justification that is usually given is that this procedure eliminates 'redundant' complexions obtained through a permutation of the particles. The issue now is, how are we to interpret this procedure? One proposal is that it shows that we should not take classical particles to be individuals, at least not on the basis of arguments involving their permutation.[254] As we shall see, the usual contrast with quantum physics is then lost. However, this proposal radically misconstrues the historical situation and, by blurring the ontological character of classical particles, blurs the distinction between classical and quantum statistical mechanics in general. Historically, the failure of extensivity and the Gibbs Paradox in general were seen as revealing a fundamental flaw in the Maxwell-Boltzmann definition of entropy and one which is corrected by shifting to quantum statistical mechanics at the theoretical level and an understanding of the particles as *quantum* in nature at the ontological level. It is in this sense that quantum mechanics, in general, has been generally taken to 'resolve' the problem.[255] In other words, the force of the proposal can be turned around: what it suggests is that the world is actually quantum in nature, as one would expect. What the exclusion of the permutations (by dividing by *N*!) is a manifestation of is *precisely* that the particles are not just indistinguishable in the classical sense. And if this aspect is incorporated into the analysis from the word go, then Gibbs' 'Paradox' simply does not arise. Thus, we take the import of the paradox to be not that we must give up an understanding of classical particles as individuals, but that we must shift to a quantum understanding of particles in general.

[253] See, e.g., Yourgrau, van der Merwe and Raw 1966, pp. 236–7.

[254] See, e.g., Saunders forthcoming p. 22, fn. 13.

[255] See, e.g., Boyer 1970.

Given, then, the permutation argument, how should we understand the individuality of the particles? We have argued that although some form of non-spatio-temporal TI may be *sufficient* for classical statistical mechanics, it is not *necessary*. The weighting assignments required to get the right (Maxwell-Boltzmann) statistics can always be obtained either by transforming to an appropriate phase space where the relevant trajectories are made apparent or by insisting that at most one particle can occupy a given space-time point. In either case, the individuality of the particles is conferred via some form of the Space-Time view.[256] Further articulation of this latter account has taken us into another whole debate about the nature of space-time itself and there we need to call a halt. Our principal conclusion in this chapter is that what we have in the classical case is a kind of underdetermination of the metaphysics of individuality by the physics: the physics by itself cannot tell us which package—TI or STI—we should prefer. As we shall now see, a much more radical form of underdetermination exists in the quantum domain, where the physics cannot tell us whether the particles are even individuals or not.

[256] There is a more general conclusion that can be drawn as well: in so far as these weights can be obtained using the field-theoretic approach, this particular obstacle to adopting a field-theoretic description of classical mechanics has been removed. This may be used to support the claim that although there are clear ontological differences between the particle and field representations, there are no observable differences between them.

# 3
# Quantum Statistics and Non-Individuality

> Quantum mechanics was born in statistics and it will end in statistics.
>
> E. Schrödinger (*in* Moore 1989, p. 416)

As Schrödinger indicates, the history of quantum mechanics in general is intimately bound to the history of quantum statistics. The fundamental framework of Planck's original work in this area was provided by Boltzmann's Combinatorial Approach, but with a subtle twist in the form of a different way of counting arrangements. It took Boltzmann's former student Ehrenfest several years to work out just what the nature of that twist was and the metaphysical implications only gradually became apparent. Putting it crudely, in quantum statistics a permutation of indistinguishable objects between states is typically not regarded as leading to a new arrangement. Hence, it was concluded, such objects should not be regarded as individuals, or at least not in the sense that they are taken to be in Maxwell-Boltzmann statistics. We shall examine this argument in more detail in the next chapter, where we will indicate how the conclusion might be resisted. Here we wish to trace out its historical emergence, as physicists grappled with the new theory.

Thus the history of quantum statistics is not only a history of the physics which underpins the philosophical conclusions regarding particle individuality, it is also a history of attempts to come to an understanding of those conclusions. As we shall see, within a year or two after the 'old' quantum theory had metamorphosed into what we now know as quantum mechanics, the untenability of retaining a 'classical' view of individuality was already being insisted upon by Bohr and Weyl among others. But perhaps the best-known proponent of the new metaphysics was Schrödinger and we shall look at the development of his view in some detail, since it will figure prominently in the formal part of our book. Finally, we shall also consider the further development

of quantum theory, as it embraced more general forms of statistics which apply to such 'exotic' entities as paraparticles. As we shall see in the next chapter, these developments will be of direct relevance for our philosophical concerns.

# 3.1 THE EARLY HISTORY OF QUANTUM STATISTICS

Determining the origins of a revolution is always a tricky business. Most commentators take Planck's famous paper of 1900[257] as marking the birth of quantum theory, although Dorling, in his presentation of the history of quantum mechanics,[258] suggested that Planck failed to understand what he had initiated and Kuhn[259] subsequently argued that the revolution actually began with Einstein's explanation of the photoelectric effect, since he took the new 'quanta' as more than mere calculational devices. In either case, Schrödinger's claim above still stands, as both Planck's and Einstein's work were intimately bound up with the application of statistics.

Like Boltzmann, Planck had grappled with the physical basis of the Second Law of Thermodynamics, but rejected Boltzmann's statistical mechanics in favour of an approach based on the foundations of electromagnetism.[260] Although this turned out to be less than fruitful in the thermodynamical context, it placed him in a excellent position from which to tackle a critical problem in late-nineteenth-century physics. This was the problem of 'black-body radiation': a 'black body' is a perfect (and hence idealized) absorber and emitter and the radiation emitted depends, not on the nature of the body, but only on its temperature. Consideration of a black body, whether theoretical or experimental (in the form of close physical approximations to this idealization), then allowed physicists to study the nature of electromagnetic radiation and in particular, the distribution of the energy over the various frequencies of the radiation. Unfortunately, however, the observed distribution did not mesh with the theoretical expressions obtained from classical electromagnetic theory. Through a process of what might broadly be described as 'phenomenological construction', Planck obtained a radiation law which was empirically adequate.[261] In order to ground this law in a micro-physical description, in an act

[257] Planck 1900, p. 202, reprinted in ter Haar 1967, p. 80.

[258] Given to students of the History and Philosophy of Science Department at Chelsea College, University of London, during the 1970s; Dorling 1967.

[259] Kuhn 1978.

[260] Kuhn 1978.

[261] For a reconstruction of Planck's work, see ter Haar 1967, pp. 3–14.

of desperation, he turned to Boltzmann's Combinatorial Approach, applied now to the oscillators of the black body.[262] However, the details of the statistics used by Planck turned out to be radically and fundamentally different from Boltzmann's.

The crucial step in Planck's derivation of his law came from thinking about how a given total energy could be distributed over $N$ linear oscillators, all vibrating with frequency ν. He divided this energy into a finite number of energy elements, $P$, which were then distributed over the oscillators. The number of ways of doing this, that is, the number of 'complexions', is given by[263]

$$\frac{(N+P-1)!}{(N-1)!P!}. \tag{3.1.1}$$

It is not immediately clear how Planck arrived at this expression although Rosenfeld has suggested that he discovered it by working backwards from his radiation law.[264]

Planck himself considered the derivation of his law to be based on only two theorems: the well-established relation between the energy density of the radiation and the average energy of the oscillators; and Boltzmann's relationship between the entropy of a system and the logarithm of the total number of possible complexions (as presented in Chapter 2). Planck split the latter into two parts:

(1) The entropy of the system in a given state is proportional to the logarithm of the probability of that state and
(2) The probability of any state is proportional to the number of corresponding complexions, or, in other words, any definite complexion is equally as probable as any other complexion.[265]

The first part simply expresses Boltzmann's crucial insight that, as Planck subsequently put it in his Nobel Prize acceptance speech, 'entropy is a measure for physical probability'. The second was regarded by Planck as the core of his whole theory and the proof of this part he took to rest ultimately on empirical grounds. He considered it to be a more elaborate and detailed expression of

[262] Klein 1970, p. 221; Kuhn 1978, p. 72ff; Pais, 1982, pp. 368–72; also see Planck in ter Haar op. cit., p. 81 and p. 83. Kuhn has been (mildly) criticized for not stressing enough the conceptual continuity between Boltzmann's and Planck's work (Cerreta 1995).

[263] Planck op. cit., p. 237.

[264] Rosenfeld 1936, p. 149; Cranfield 1968, p. 69; Kuhn op. cit., pp. 100–1; for a discussion of the source of this expression see Kuhn ibid., p. 282.

[265] Planck in ter Haar op. cit., p. 87.

the hypothesis that the energy of the radiation is randomly distributed over the various partial vibrations of the oscillators.

Thus the statistical component of Planck's work followed Boltzmann's Combinatorial Approach in broad outline but differed from it in significant respects. First of all, whereas Boltzmann let the size of his energy elements decrease to zero, as we have seen, Planck did not, but left them as finite and related to the frequency via the famous expression $\varepsilon = h\nu$.[266] Secondly, and more importantly from our point of view, Planck's combinatorial expression above is very different from Boltzmann's. Planck considered, at least implicitly, only the number of energy quanta assigned to each oscillator and not *which* quanta were possessed by *which* oscillators. Thus the numerator of his expression gives simply the total number of ways of arranging the quanta and oscillators, regarded as $N + P - 1$ distinct elements (we shall discuss the significance of the $-1$ when we come to consider Ehrenfest's analysis of this result).

The division by $P!$ implies that permutations of the quanta are not regarded as giving rise to new, countably different arrangements. As we have already indicated, it was that which led to the conclusion that the quanta should be regarded as not only indistinguishable but also unlabelled and devoid of individuality.[267] Thus whereas Boltzmann considered the distribution of indistinguishable but individual atoms over energy states or cells in phase space, Planck was effectively concerned with the distribution of indistinguishable but *non-individual* quanta over oscillators.[268] The non-classical nature of Planck's expression and the questions surrounding its interpretation became a focus of early criticism, particularly from Ehrenfest, as we shall see, who undertook to expose the conceptual foundations of the new quantum theory.

Finally, we recall that Planck's second sub-theorem above, regarding the probability of a state, is simply a restatement of the crucial assumption that equal *a priori* weights should be assigned to the various complexions. It was again Ehrenfest who subjected this assumption to critical examination and who first discussed the way in which the weighting assignments of classical statistical mechanics had to be revised in the light of the new physics.

[266] It is not clear whether Planck regarded his energy quanta as having real physical significance or as merely convenient mathematical artefacts; see Kuhn op. cit., and Dorling 1967.

[267] See for example, Pais op. cit., p. 370 or Kuhn op. cit., p. 101.

[268] Cranfield remarks that "… in Planck's method, energy elements were the distributed quantities rather than oscillators whose distribution over phase space would have been the analogue of Boltzmann's approach" (Cranfield op. cit., p. 70).

Ehrenfest's first thoughts on this subject were set down in his 1905 paper in which he expressed his puzzlement over the exact nature of the relationship between Planck's work and Boltzmann's Combinatorial Approach.[269] In particular, he noted that Planck's choice of those states to be given equal *a priori* probabilities was different from Boltzmann's. This difference arises from the different choice of events to be counted: whereas Boltzmann attached equal *a priori* probabilities to each region of phase space, Planck assigned them to each energy distribution.[270] Ehrenfest was well aware that a definite hypothesis concerning such assignments was crucial to the Combinatorial Approach and six years later he presented a generalized weight function suitable for the new statistics.[271]

1905 also saw the next major advance in the history of quantum physics, with the publication of Einstein's famous paper in which he proposed the hypothesis of independent energy quanta in radiation.[272] Interestingly, the central argument of this paper is based, not on Planck's law, but on the classical expression for the distribution of radiation energy (Wien's law, to be specific) which is strictly empirically inadequate. Nevertheless, Einstein was able to obtain the correct results because he treated his quanta as Maxwell-Boltzmann particles instead of—what came to be known as—bosons.[273] It is precisely in the limit of the classical expression that the latter behave like the former. It turns out that Einstein's result can also be derived using the correct distribution law—namely Planck's—and the correct quantum statistics—namely, Bose-Einstein, as one would nowadays expect.[274] In other words, if the quanta are treated as statistically independent, an empirically inadequate expression is obtained. The fact that Planck's law is actually the correct one therefore suggested that the light quanta were not statistically independent. This point, and the related question of the nature of the interdependence between quanta, was subsequently the subject of much discussion by Ehrenfest and others, as we shall see.

[269] Ehrenfest 1905, p. 1301; 1959, p. 88.

[270] Cranfield op. cit., p. 70.

[271] Rayleigh and Jeans were also puzzled by Planck's derivation (Rayleigh 1905, p. 547; Jeans 1905, p. 293). In particular Jeans argued that Planck's combinatorial expression violated the classical probability calculus because he had not specified the population from which such probabilities could be calculated. Such a population could not be introduced with an *a priori* weighting assignment consistent with Planck's arguments.

[272] Einstein 1905; also in ter Haar op. cit.

[273] Cranfield op. cit., p. 125. Thus one could argue that, contrary to Kuhn's claim, this 1905 work should not be regarded as the vanguard of the quantum revolution precisely because it does not incorporate the most revolutionary aspect of Planck's work.

[274] op. cit.

By the following year Einstein had realized that Planck's theory pre-supposed the existence of light quanta[275] and in 1907 he noted that if this theory were correct, then energy values other than those given by $h\nu$ are not accessible to the oscillators.[276] His analysis of energy fluctuations in black body radiation in 1909 produced an expression for the mean square energy fluctuation which was the sum of two terms.[277] The classical wave theory of light would give the second term, associated with wave phenomena, only, and the first, the particle term, corresponded to " … independently moving pointlike quanta with energy $h\nu$"[278]

This is the first intimation of 'wave-particle duality' but it is interesting that Einstein still regarded the quanta as statistically independent here. Their non-classical interdependence could then be explained by referring to their wave-like aspect. Regarded as crests in a system of waves, quanta would become properties of the whole system and the probability of a quantum existing in a particular state would then be dependent on the other quanta in the system. As we shall see, Einstein subsequently justified the counting procedure that has come to be associated with his name and that of Bose, by referring to de Broglie's suggestion that particles are analogous to crests in a system of waves; a suggestion that was subsequently taken up by Schrödinger and elaborated into his so-called 'wave mechanics'.

The second point to note concerning Einstein's statement above is that he referred to the light quanta as point-like. It is clear that by 1909 he was thinking of quanta as particles.[279] This view was reinforced by his work on spontaneous and induced radiative transitions in the course of which he derived the result that a light quantum carries momentum $\frac{h\nu}{c}$ [280] Thus there emerged the concept of a photon as a particle carrying both energy and momentum.

Planck went on to present a rearranged version of his 1900 derivation[281] and introduced an alternative approach involving a phase-space description of the equiprobable regions accessible to an oscillator. These regions were elliptical rings of area $h$ on the energy hypersurface, within which lay the oscillators, and consideration of the distribution of the oscillators over these areas then allowed him to compute directly the number of complexions corresponding to a given state. This formed the basis for a quantized statistical treatment of the distribution of entities—particles or resonators—over certain regions

[275] Einstein 1906, p. 199.

[276] Einstein 1907, p. 150.

[277] See Pais op. cit., pp. 402–10 for a good discussion of this analysis.

[278] Einstein 1909, pp. 815–17.

[279] Pais op. cit., p. 403.

[280] Einstein 1916, p. 47 and Einstein 1917, p. 121.

[281] Planck 1906, ch. 4.

of phase space, which was much more akin to the classical combinatorial procedure. Quantal results were ensured, however, by the fixed size given to these regions as determined by *h*.[282]

This method was subsequently generalized and a natural unit for phase extension of *h* for each degree of freedom established. This meant that each cell in phase space could be given the volume *h*, thus giving an absolute value for the entropy.[283] These results implied that the classical assignment of equal *a priori* weights to equal volumes of phase space had to be abandoned and only certain regions of this space given non-zero weight. Thus for Planck's oscillators these regions were those particular ellipses of constant energy whose enclosed areas were integral multiples of *h*. As we have said, it was Ehrenfest who was virtually alone in recognizing that the basis of Boltzmann's proof of the Second Law had been lost in principle as soon as physics had followed Planck in abandoning the classical weighting assignment[284] and he went on to present a suitably generalized weight function for quantum statistics which did not rely on the notion of an oscillator.[285] This was part of a move to a more general conception of quantum statistics which was applicable to both light quanta and material particles.

Returning to Planck's distribution law, Ehrenfest showed that it could be derived without recourse to oscillators as well and also that the entropy function that was obtained was common to Boltzmann and Planck.[286] He thus concluded that the source of the difference between classical and quantum statistical mechanics must lie, *not* in the entropy equations themselves *but in the imposed constraints*, that is, in the quantization requirement. Subsequently, he examined the difference between Planck's quantized oscillators and Einstein's light quanta. The prevailing opinion at the time was that the main difference lay in the separate existence of quanta in empty space. However, Ehrenfest pointed out that the statistical independence of quanta was another source of contention, with Einstein affirming and Planck denying it. As he noted, Planck had not assumed this in his derivation and it led, in fact, to Wien's law, which Einstein *had* used. This problem of the interdependent statistical behaviour of Planck's quanta clearly troubled Ehrenfest and not only was it

[282] Thus, as Cranfield has noted, although the results of classical statistics are invariant with respect to a change in the volume of the cells in phase space, those of quantum statistics are not.

[283] Via the Sackur-Tetrode Equation, mentioned in Chapter 2.

[284] See Ehrenfest's letter to Bohr, given in Klein op. cit., p. 283.

[285] Ehrenfest 1914, p. 657; 1959, p. 347.

[286] Ehrenfest 1906, p. 528.

an issue to which he himself would return, as we shall see, but it also lay at the centre of an early debate over the individuality of quanta.

Like Ehrenfest, Natanson was also interested in the differences between Planck's and Einstein's conceptions. In particular he examined the assumptions underlying Planck's consideration of the distribution of energy elements over 'receptacles', and identified three different 'modes of distribution', depending on whether the quanta or receptacles or both were regarded as individuals.[287] Thus he used the term 'mode of distribution' for the correlation of energy elements with receptacles in which the distribution is characterized only by the number of receptacles containing a given number of quanta. In this case no account is taken of the possible individuality (or 'identifiability') of receptacles or energy elements. However, if the former are regarded as individuals then every 'mode of distribution' branches out into a number of 'modes of collocation', which specify the number of energy elements in each individual receptacle. This, he argued, corresponded to Planck's microstate. If energy elements are also regarded as individuals then each 'mode of collocation' splits up into a number of 'modes of association', which associate individual energy elements with individual receptacles. This corresponds to Boltzmann's notion of a complexion.[288]

As Natanson noted, the thermodynamic probability of a given mode of distribution depends on whether all modes of collocation or all modes of association are regarded as equally probable. Whereas Einstein took the latter view, Planck took the former and thus, according to Natanson, implicitly regarded his receptacles, the oscillators, as individuals but not his energy elements, the quanta. The odd terminology is an attempt to distinguish the meaning of the term 'complexion' as used in quantum and classical statistics. In the former, a complexion specifies only how many elements are in each receptacle and this is called a mode of collocation. A permutation of two elements between receptacles does not lead to a new mode of collocation. In the latter, a complexion specifies which individual element is in which individual receptacle, and this is termed a mode of association. A permutation of elements between receptacles *does* lead to a different mode of association.

Joffe also attempted to modify Einstein's idea of quanta so as to reconcile it with Planck's law,[289] and a controversy subsequently arose between Krutkow

[287] Natanson 1911, p. 659.

[288] An interesting, although rather broad, account of these developments is given in Kastler 1983.

[289] Joffe 1911, p. 53.

and Wolfke over just this question. Krutkow, following Ehrenfest, demonstrated that if classical statistics are used, then the assumption of independent quanta leads to Wien's law. Planck's law can only be obtained, he argued, if this assumption is abandoned. Wolfke, however, argued that one must distinguish between two meanings of the word 'independent'. If the quanta *exist* independently then, he claimed, Planck's law is obtained, but if, in addition, the quanta are regarded as *spatially* independent then Wien's law will result.[290] Thus he concluded that Einstein had implicitly assumed the quanta to both exist independently and to be spatially independent whereas Planck has assumed only the former, suggesting that some sort of spatial correlation had to exist between the quanta. It is in these debates and the work of Ehrenfest that we see the first explorations of the metaphysical implications of quantum statistics. As we shall see, these concerns regarding individuality, re-identifiability and independence were to arise again and again in the development of the field.

Further light was shed on these questions by an important paper of Ehrenfest and Kammerlingh Onnes,[291] which gave an intuitive way of understanding Planck's combinatorial formula. Ehrenfest had not been satisfied with the standard derivations given in the textbooks as they appealed to proof by induction and offered no insight into the peculiar structure of the result. Thus he set out to provide his own.

The problem he and Kammerlingh Onnes considered was to find the number of ways in which $P$ objects could be placed in $N$ containers where only the number of objects in each container is of importance. Two distributions were called identical when corresponding containers, that is, oscillators, in each distribution possessed the same number of objects, that is, energy elements. A distribution was represented symbolically thus

$$\Pi\,\varepsilon\varepsilon\varepsilon\varepsilon\;\circ\;\varepsilon\varepsilon\;\circ\;\circ\,\varepsilon\,\Pi \tag{3.1.2}$$

where the Π's are the 'fixed boundaries', ε represents the energy units, and the *o*'s separate these units in successive oscillators, numbered from left to right.[292] With general values of $N$ and $P$ the symbol will contain $\varepsilon P$ times and *o* $N-1$ times, and they asked how many different symbols for the distribution could

[290] Wolfke 1914, pp. 133 and 363.

[291] Ehrenfest and Kammerlingh Onnes 1914, p. 870 and 1959, p. 353; see Klein 1970, pp. 255–7 and Ehrenfest 1959, pp. 47–8.

[292] This notation is confusing in that the symbol ^ has the dual significance of representing both the separation of quanta in successive oscillators and also that an oscillator possesses no quanta. A clearer way of writing it is ‖εεεε|εε‖, where the dividers separate the energy symbols for different oscillators; Klein 1970, p. 256.

be formed from the given number of ε and *o*. The answer was

$$\frac{(N+P-1)!}{(N-1)!P!}$$

(3.1.3)

This is derived as follows. There are $N + P - 1$ quantities ε and *o* which, if regarded as distinguishable, can be arranged in $(N + P - 1)!$ ways between the ends Π. However, the εs and *o*s can be permuted among themselves $P!$ and $(N - 1)!$ times respectively and two distributions differing only by such permutations are identical. Therefore as there are $P!(N - 1)!$ symbols corresponding to these identical distributions one must divide $(N + P - 1)!$ by $P!(N - 1)!$ to obtain the number of distinct distributions. The result is Planck's combinatorial formula above.

This derivation allowed some important conclusions to be drawn regarding the status of Planck's quanta and their relationship with Einstein's. It confirmed Ehrenfest in his old opinion that the energy elements ε, and hence also Planck's quanta, were merely a formal device, imbued with no more physical significance than the divider elements.[293] The failure to realize this, he believed, had led to the mistaken interpretation that these quanta were mutually independent and identical to Einstein's. The difference between the two was demonstrated by comparing their different statistical behaviour. It was noted that the number of ways of distributing a number $P$ of Einstein's light quanta, regarded as independent, over $N_1$ and then over $N_2$ cells in space stand to each other in the ratio

$$N_1^P : N_2^P$$

(3.1.4)

If Planck's quanta were also regarded as mutually independent, then in passing from $N_1$ to $N_2$ oscillators, the number of possible distributions would increase in the same ratio. This is clearly not true, as Planck's formula gives

$$\frac{(N_1-1+P)!}{(N_1-1)!P!} : \frac{(N_2-1+P)!}{(N_2-1)!P!}$$

(3.1.5)

According to Ehrenfest and Kammerlingh Onnes, the explanation was simple: Einstein's quanta could be regarded as existing independently of each other, whereas Planck's could not[294] and were no more than a formal device:

> The real object which is counted remains the number of all the different distributions of $N$ resonators over the energy grades 0, ε, 2ε,… with a given total $P$.[295]

[293] Ehrenfest and Kammerlingh Onnes op. cit., p. 872.

[294] We recall Wolfke's distinction above.

[295] op. cit., p. 873.

Finally, they gave an example of the quantal reduction in the number of possible arrangements which illustrated the classical nature of Einstein's quanta and the non-classical behaviour of Planck's.

The claim, then, was that if Planck's quanta were to be considered as statistically independent then the traditional classical counting procedure should be applied rather than Planck's own.[296] Since the number of distributions obtained by Planck differed from that obtained by treating the quanta as independent the entropy change in any appropriately specified process would also have to be different. It is ironic that although the above derivation is now almost universally given as an intuitive underpinning of Planck's procedure, its original purpose was to argue that this procedure cannot be correct if the quanta are to be independent. This was also the substance of Ehrenfest's criticism of Einstein's 1924 paper proposing Bose-Einstein statistics, as we shall see.

It follows, then, in a reciprocal manner, that if Planck's procedure *is* correct then the quanta cannot be regarded as statistically independent and must exhibit some sort of non-classical correlation. Furthermore, the fact that a permutation of Einstein's quanta gives a new distribution whereas a permutation of Planck's does not implies that the former can be regarded as individuals whereas the latter cannot. Klein has remarked that the blurring of the concept of particle when it comes to light quanta was already implicit in Einstein's results for energy and momentum fluctuations in black body radiation.[297] These results furthered the identification between Einstein's and Planck's quanta because they effectively allowed Einstein to explain quantum phenomena in terms of the wave-like aspect associated with his, individual, quanta, whereas Planck effectively had to appeal to the non-classical correlations between his, non-individual, quanta.

## 3.2 THE LATER HISTORY OF QUANTUM STATISTICS

From about 1910 onwards attention shifted increasingly towards the problems of atomic structure and other questions, but the publication of Bose's work in 1924, and the subsequent elaboration of its consequences by Einstein, rekindled many of the old arguments concerning the appropriate statistics.[298]

[296] Klein 1970, p. 256.

[297] Ibid., p. 257.

[298] For further details on the history of Bose-Einstein statistics, see Bergia 1987.

In this paper, Bose presented a derivation of Planck's law on the basis of Einstein's light quantum hypothesis using the traditional combinatorial formula.[299] As is well known, he asked Einstein to translate his work and arrange for its publication. In the accompanying letter he emphasized the division of phase space into cells of volume $h$ as a basic assumption of his approach, clearly indicating its ancestry. The quanta were regarded as particles which were localizable in space and Bose considered their distribution over the cells of phase space. It was in determining this distribution that he deviated from the traditional line and adopted Planck's 1900 approach, in specifying the distribution by the *numbers of cells containing each possible number of quanta* rather than by which quanta are in each of the cells. Thus he employed a quantum rather than a classical characterization of the events to be counted: cells occupied by the quanta.

However, Bose then used a form of the traditional Boltzmann combinatorial formula, giving the number of ways $W$ of distributing $A^s$ cells over $N^s$ quanta as

$$\Pi_s \frac{A^s!}{P_0^s! P_1^s! \ldots} \tag{3.2.1}$$

where $P_i^s$ is the number of cells containing $i$ quanta. Indeed Bose followed the classical combinatorial approach very closely, but replaced everywhere 'particles' by 'cells'[300]

The view that this work represents a significant departure from traditional Boltzmann statistics, as modified by Planck in 1906, is therefore perhaps a little simplistic.[301] Bose did depart from the traditional line as regards what was taken as a countable event, although the departure was hardly strikingly original, merely a return to Planck's characterization of 1900, but the combinatorial formula employed and his whole procedure in general were located entirely within the traditional approach. It is worth remarking that, just as Boltzmann's procedure implied that his atoms were statistically independent, so Bose's juxtaposition of the classical combinatorial expression with the quantal characterization of countable events implied the statistical independence of the cells. The statistical independence of the quanta had vanished. As we shall see, this aspect of Bose and Einstein's work was seized upon by Ehrenfest as a perpetuation of Planck's earlier mistake. Finally we emphasize

[299] Bose 1924, p. 178 rep. in Boorse and Motz 1966, p. 1013; also see Pais op. cit., p. 423ff.

[300] Thus $A^s$ is not the number of particles as in Boltzmann's work but the number of *cells*.

[301] Wessels 1977, p. 315.

again that by counting the cells containing the quanta, rather than the quanta themselves, and by asking how many quanta are in a cell rather than which quanta are in a cell, Bose could be understood as implicitly regarding the quanta as devoid of individuality in the classical sense.[302]

Shortly after he had arranged for the publication of Bose's paper, Einstein presented a paper of his own in which he applied Bose's methods to a gas of material particles. This paper and the two which followed it laid down the foundations of the quantum theory of the ideal gas embodying what is now called Bose-Einstein Statistics. In his first paper Einstein used exactly the same combinatorial formula and techniques as Bose, suitably modified to take into account the finite mass of the gas atoms and their fixed number, and employed a similar characterization of countable events, with 'quanta' replaced by 'gas atoms'.[303] Thus Einstein followed Bose in using the traditional form for the expression for the number of ways of distributing $A^s$ cells over $N$ particles, given above, together with the 'Planck 1900' characterization of the events to be counted. The thermodynamic properties of Einstein's gas were consequently more complicated than in the classical case but tended to support his theory. It predicted a value for the entropy at high temperatures, for example, which was equal to that given by the Sackur-Tetrode equation, mentioned earlier.[304] Einstein also showed that at temperatures approaching absolute zero the entropy approached zero for all values of the volume, demonstrating that his quantum gas also satisfied Nernst's heat theorem. Further support was to come several years later with the experimental verification of the famous low-temperature degenerate behaviour of liquid helium.[305]

This behaviour was the subject of one of Einstein's letters to Ehrenfest, where he wrote "From a certain temperature on, the molecules 'condense' without attractive forces, that is, they accumulate at zero velocity",[306] and he wondered how true his theory was. It is interesting to note that Einstein appears to have ruled out the possibility of the non-classical condensation phenomena being accounted for in terms of peculiar interatomic forces. In a reply, sometime that same year, Ehrenfest criticized this work on the grounds that the gas

[302] See Klein 1964, p. 29.

[303] Einstein 1924, p. 261.

[304] Ehrenfest and Trkal 1920, p. 162, also Ehrenfest 1959, p. 414.

[305] The application of Bose-Einstein statistics to the superfluid behaviour of liquid helium was proposed by London in 1938 (London 1938). London wrote that "… in the course of time the degeneracy of the Bose-Einstein gas has rather got the reputation of having only a purely imaginary existence" (1938a, p. 644). Of course, in applying the statistics of an ideal gas to a liquid, certain idealizations had to be made; for details of what was involved see Bueno, French and Ladyman 2002.

[306] Pais op. cit., p. 432.

molecules were not statistically independent.[307] This criticism is pivotal to an understanding of the development of Einstein's work in this area, as we shall now see.

Einstein's second paper was published in 1925 and dealt first of all with the above condensation phenomena.[308] He then went on to consider Ehrenfest's objection and accepted that it was entirely correct. In the course of the discussion of this issue Einstein gave a combinatorial formula and a characterization of countable events which were completely different from those given in the previous year. Thus he now wrote

$$W = \Pi^{s} \frac{(N^s + A^s - 1)!}{(A^s - 1)! N^s!} \tag{3.2.2}$$

where $A^s$ is the number of cells and $N^s$ the number of particles. If certain trivial substitutions are made, then Einstein's expression is identical to the product over all frequencies of Planck's. However, Einstein now took a distribution to be characterized, in a classical manner, by the number of particles in each available cell, and counted the number of ways in which the particles could be distributed over the cells. This is a combination which is the reverse of the one used in 1924.

It is surprising that Ehrenfest directed his criticism not at this work as would perhaps be expected, but at the 1924 paper which explicitly used a version of the traditional combinatorial formula. To what, then, was Ehrenfest objecting? Clearly it was to the characterization of countable events used in 1924 which, as we have noted, implied that the cells were statistically independent but the particles were not. In other words, just as Ehrenfest criticized Planck for the statistical interdependence inherent in his combinatorial formula of 1900, he now criticized Einstein for the same thing, manifested in the characterization of what was to be counted. Einstein admitted the truth of this objection and attributed the statistical dependence to some kind of mutual interaction between the particles. Thus, referring to his expression above, he wrote, "[t]he formula, therefore, expresses indirectly an implicit hypothesis about the mutual influence of the molecules of a totally new and mysterious kind"[309] However, it was absolutely crucial that this statistical dependence was not eliminated by the 1925 reworking of the theory, despite Ehrenfest's

[307] Klein op. cit., p. 31.

[308] Einstein 1925.

[309] Einstein 1925; trans. in Duck and Sudarshan 1997, pp. 91–2.

objections because, as Einstein obviously realized, it lay at the very heart of the condensation phenomenon to which he attached so much importance. But if the particles were now regarded as individuals, in what did this dependence lie?[310] The answer is that it lay in the wave-like aspect which Einstein now attributed to his gas atoms, following de Broglie. As is very well known, this suggestion was subsequently developed by Schrödinger into his theory of wave mechanics and the wave-like nature of material particles became a way of understanding their lack of statistical independence.

This new form of gas statistics provoked a number of responses.[311] Planck, for example, presented his own conservative reply to Einstein and derived the entropy on the basis of his 1906 work, employing the N! division in order to avoid counting 'redundant' complexions formed by the permutation of two atoms.[312] This procedure was justified on the grounds that such a permutation produced no change in the state of the gas.[313] Unlike Einstein, Planck was deeply sceptical of any strange statistical interaction between the atoms and thus his theory did not predict any degenerate behaviour at low temperatures.

The procedure of dividing the number of possible arrangements—obtained by implicitly regarding the atoms as statistically independent individuals—by $N!$ in order to obtain the 'correct' quantal count had been the subject of an earlier dispute between Planck and Ehrenfest.[314] Planck had vigorously defended the division, on the above grounds, while Ehrenfest had argued that it was *ad hoc* and simply not cogent. These arguments had a big impact on Schrödinger, who believed that one could only divide by $N!$ when the gas was in the condensed state in which the atoms were virtually held fixed, so the permutation number $N!$ becomes physically meaningful and the atoms become 'identifiable'.[315] Identifiability, in this case, is conferred via a spatio-temporal 'fix'.

A year later Schrödinger reconsidered this problem in the light of Bose and Einstein's work and again attacked Planck's justification for the $N!$ division on the grounds that *the molecules were either individuals or not* and the theory should be constructed accordingly, without first assuming they were and then 'correcting away' the resulting multiplicity.[316] Thus he suggested a holistic

[310] Thus the analysis given by Brush, for example, is over simplistic (Brush 1983, p. 131).

[311] Klein 1964, p. 31.

[312] Planck 1925; see Hanle 1977, p. 177 for a good discussion of these responses.

[313] What Planck was insisting on here was the use of what Gibbs called the 'generic' rather than the 'specific' phase (Gibbs 1902, ch. l5).

[314] Ehrenfest and Trkal 1920; 1921, p. 609; Planck 1921, p. 365; 1924, p. 673.

[315] Schrödinger 1924, pp. 41–5.

[316] Einstein 1925, p. 434.

view which attributed quantum states not to individual gas atoms but to the body of the gas as a whole. However, Schrödinger could not find a way to carry out such a programme in a physically plausible way. We shall return to this issue, since it is one of the criticisms of the 'orthodox' treatment of quantum statistics that it labels both particles and states, effectively treating the former as individuals, and then constructs a wave-function for the entire assembly through permutations of the particle labels, which in turn is taken to imply that the original individuality of the particles has somehow been lost. A metaphysically less convoluted approach—or so the criticism continues—would be not to label the particles or treat them as individuals right from the start, as in the case of Quantum Field Theory. It is ironic that this criticism has been levelled against what is seen as the 'Schrödinger formulation' when, as we have just indicated, Schrödinger himself held a similar view and, as we shall also see, advocated the move to a quantum field theoretic formulation. As Klein has noted,

> … the same impasse which blocked the understanding of the statistics of photons was at least the cause of a detour in the statistics of atoms and molecules. In both cases the classical concept of the particle was at fault, since non-interacting classical particles are necessarily independent.[317]

The difficulties in adopting a more holistic approach at that time forced Schrödinger to amend his approach and in 1926 he applied Boltzmann's combinatorial procedure to a gas considered, significantly, as a collection of de Broglie matter waves.[318] This gave a theory broadly similar to Einstein's but which again ruled out the possibility of the condensation effect predicted by the latter.

In 1928 the He I–He II phase transition was discovered and this was subsequently interpreted as an example of B–E condensation, but not until 1938.[319] Thus it was not the experimental verification of a crucial prediction which decided between Einstein's gas theory and the rival approaches of Planck and Schrödinger but rather, as we shall now see, the accommodation of this theory within a self-consistent theoretical framework by Heisenberg and Dirac.[320]

[317] Klein 1964, p. 55.

[318] Schrödinger 1926; also see Pais op. cit., pp. 438–9.

[319] Brush op. cit., p. 172–203. See fn. 56 above.

[320] Planck continued to propound his approach until 1926 but appears to have realized that he had strayed away from the mainstream of quantum statistical mechanics.

With the construction of the 'new' quantum mechanics from 1925 to 1927, three different authors independently applied the theory to the statistical mechanics of indistinguishable particles.

Fermi's approach was significantly different from Bose and Einstein's in that he attempted to formulate a theory of the ideal gas which assumed the validity of Pauli's Exclusion Principle[321] for gas atoms. Thus he obtained an expression for the number of arrangements of $N_s$ molecules distributed over $Q_s$ states, subject to the constraint that not more than one molecule could be in any one state. The Boltzmann relation and standard thermodynamics then gave the equation of state for such a gas, now known as the Fermi gas formula. On this basis Fermi also derived the Sackur-Tetrode equation, demonstrating that his gas would exhibit the correct behaviour at high temperatures. With the publication of this work there were then two very different theories of the ideal gas, each embodying a different form of statistics. Shortly afterwards Heisenberg explicated the connection between these two forms and the symmetry characteristics of states of systems of indistinguishable particles.

In his first paper, published in June 1926, Heisenberg showed that two indistinguishable systems which were weakly coupled, always behaved like two oscillators for which there were two sets of non-combining states.[322] Thus he demonstrated that the eigenfunctions of one system were symmetric in all the coordinates whereas those of the other were anti-symmetric. The fact that the two sets of states were not connected then followed from the symmetry of the Hamiltonian of the system, under a particle permutation. An example of such systems, according to Heisenberg, were the two electrons in the helium atom and in a subsequent work he investigated more fully the theory of such two electron atoms using Schrödinger's wave mechanics.[323] The conclusion he reached was that only those states whose eigenfunctions are anti-symmetric in their electron coordinates can arise in nature.[324]

The final element of what has become the standard understanding of quantum statistics was completed by Dirac, who had read Fermi's paper but claimed to have forgotten it and had not seen Heisenberg's at all, although

[321] Fermi 1926a, p. 145 and 1926b, p. 902. Rasetti's introduction to Fermi's collected papers has a good discussion of the genesis of this work; Fermi 1962, p. 178; Pauli 1925, p. 765.

[322] Heisenberg 1926a, p. 411.

[323] Heisenberg 1926b, p. 499.

[324] For further details see Miller 1987, pp. 311–13. It is in this work that Heisenberg introduced group-theoretical methods into quantum mechanics, independently of Wigner. However, as he acknowledged, his analysis was not sufficiently general and it was left to Wigner, together with Weyl and von Neumann, to further develop these methods. And as Wigner and von Neumann noted, the application of such methods rests on two fundamental features: the equivalence of all directions in space and the "equality" of every electron (von Neumann and Wigner 1928).

he mentioned it in a note added in proof. Thus his own 1926 paper represents a more or less independent effort which went further in setting the two forms of statistics and the corresponding theories of the ideal gas within their currently accepted theoretical context.[325] Indeed, it is in this paper that the distinction between the two forms and their relationship to the Exclusion Principle—something which was still confused in Heisenberg's work—was fully clarified. He began with the fundamental requirement that the theory should not make statements about unobservable quantities and noted that it then followed that two states which differed only by the interchange of two particles, and which were therefore physically indistinguishable, must in fact be counted as only one state.[326] This in turn implied that out of the set of possible two-particle eigenfunctions there were only two which satisfied the conditions that the eigenfunction should correspond to both of the above states and should be sufficient to give the matrix representing any symmetric function of the particles, these two being the symmetrical and the anti-symmetrical eigenfunctions. Although Dirac noted that the theory as it then stood was incapable of deciding which of these two actually applied in nature,[327] he showed that the Exclusion Principle followed quite naturally from the anti-symmetric form and remarked that "[t]he solution with symmetrical eigenfunctions … allows any number of electrons to be in the same orbit so that this solution cannot be the correct one for the problem of electrons in an atom".[328] In other words, as the theory cannot say which solution is correct, extra-theoretical considerations had to be appealed to.

This framework was then applied to the ideal gas, and Dirac identified the symmetrical eigenfunctions with Bose-Einstein statistics, taken to be applicable to light quanta, and the anti-symmetrical with what are now known as Fermi-Dirac statistics, regarded as applying to electrons in atoms and gas molecules. The equation of state of an ideal gas was then derived on the assumption that the solution with anti-symmetrical eigenfunctions is the correct one, so that not more than one molecule can be associated with each de Broglie matter wave.[329] The set of such waves associated with the molecules was then divided into a number of subsets such that the waves in each subset are associated with molecules of about the same energy. Assuming that equal *a priori* weights are assigned to all stationary states of the assembly, the probability of a distribution in which $N^s$ molecules are associated with the $A^s$

[325] Dirac 1926.

[326] Ibid., p. 667.

[327] Ibid., p. 669.

[328] Ibid., p. 670.

[329] Ibid., p. 672.

waves in the s'th set is given by[330]

$$W = \prod^{s} \frac{A^s!}{(A^s - N^s)!N^s!}$$

(3.2.3)

Boltzmann's relation was then used to give the entropy and the equation of state was obtained very straightforwardly.

Dirac's combinatorial formula, above, is simply a suitably amended version of the well-known expression for the number of ways in which *m* objects can be selected from a set of *n* objects; that is, the number of combinations of *n* things taken *m* at a time.[331] Thus it gives the number of ways in which $N$ molecules can be selected from a set of $A$ molecules associated with $A$ waves, or the number of possible combinations of $A$ waves taken $N$ at a time. We can understand Dirac's formula in the following way: if we have $A^s$ waves and $N^s$ molecules then the number of ways of associating these molecules with the waves, such that no wave is associated with more than one molecule, is given by

$$\frac{A^s!}{(A^s - N^s)!}$$

(3.2.4)

However, a permutation of the molecules does not lead to a new arrangement and so this result is divided by the total number of arrangements formed by such permutations, $N!$, giving

$$\frac{A^s!}{(A^s - N^s)!N^s!}$$

(3.2.5)

We repeat this procedure for each set of de Broglie waves and the probability of a particular distribution is then given by the product of the above over all sets. In other words, Dirac's formula is simply a version of Planck's non-traditional 1900 formula, suitably amended to take account of the Exclusion Principle. As Dirac was concerned with the distribution of particles over de Broglie waves, or states, he implicitly adopted a classical characterization of particles distributed over states as the events to be counted.

With the realization that there were now three kinds of statistics—Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac—several papers were published

[330] Dirac 1926, p. 673.

[331] Margenau and Murphy 1956, p. 432.

exploring the relationship between them. Fowler was the first to construct a general form of statistical mechanics embracing all three types.[332] Ehrenfest and Uhlenbeck tackled the question as to whether B-E or F-D statistics are *necessarily* required by the formalism of quantum mechanics, or whether there were areas in which classical statistics were still valid.[333] They concluded that it is the imposition of symmetry requirements on the set of all solutions of the Schrödinger equation for an assembly of particles, obtained by considering the permutations of all the particles among themselves, which produce the symmetric and anti-symmetric combinations and hence give rise to B-E and F-D statistics respectively. If no constraints are imposed then Maxwell-Boltzmann statistics are the most appropriate form to use. Thus, they demonstrated that the quantum formalism does not, by itself, necessarily imply one or other of the two forms of quantum statistics.

We have noted that the non-classical statistical dependence of the particles manifested in quantum statistics could be 'explained' by reference to the de Broglie matter waves associated with each particle. In these terms, Uhlenbeck contended that Schrödinger had shown in 1926 that Einstein's gas theory could be obtained by considering the gas either as an assembly of particles and applying B-E statistics or as a system of standing de Broglie waves and applying classical M-B statistics.[334] He used wave mechanics to extend this discussion into a detailed analysis of the various interpretations of the three forms of statistics and their different areas of applicability. The conclusion he reached was that the lack of independence of quantal particles was introduced in a natural way in the appropriate wave interpretation. This led Ehrenfest to remark, "[O]ne must distinguish between identity and independence, between the particle and the wave picture".[335]

It is also worth noting that these developments took place entirely within the Combinatorial Approach and therefore were capable of treating equilibrium situations only. The neglect of Boltzmann's alternative *H*-Theorem Approach ended in 1928 with Nordheim's attempt to construct a theory which would not only give all the results derived previously, but also expressions applicable to phenomena associated with non-equilibrium states.[336] However, his conclusion that he had constructed a theory of quantum statistics based purely on kinetic arguments is not strictly correct, as combinatorial initial conditions

[332] Fowler 1926.

[333] Ehrenfest and Uhlenbeck 1927.

[334] Uhlenbeck 1927. This is not strictly correct, as we have seen, because Schrödinger's approach, using matter waves, did not give rise to the condensation phenomena.

[335] See Klein op. cit., p. 58.

[336] Nordheim 1928.

were introduced in order to obtain the correct quantum behaviour. Thus he wrote,

> The prohibition of certain states of motion, characteristic for quantum statistics, emerges here not out of the law of motion but out of the choice of the initial state … of a proper wave group. Thus, for instance, it is prohibited in the Fermi-Dirac statistics to choose an initial distribution, the density in the phase space of which is greater than one particle per $h^3/m^3$.[337]

In other words the results of quantum statistics can only be obtained if certain states are regarded as inaccessible and a particular initial state selected. Once this choice has been made, the laws of quantum mechanics can only ensure that there are no transitions to the forbidden states. Thus, on Nordheim's own admission, the *H*-Theorem alone cannot give the appropriate statistics; some external, non-kinetic constraint, such as the symmetry restrictions imposed upon the wave functions, must be imposed. Similarly, Ornstein and Kramers' attempt to derive Fermi-Dirac statistics from purportedly purely kinetic arguments[338] was based on the Exclusion Principle which, again, can be regarded as a non-kinetic initial restriction on the set of accessible states.[339] We shall return to this issue of the role of state accessibility requirements in the next chapter.

## 3.3 THE IMPACT OF QUANTUM STATISTICS: QUANTUM NON-INDIVIDUALITY

As we have seen, it was perceived quite early in the history of quantum mechanics that quantum statistics raised a number of fundamental issues regarding the nature of the quanta and their independence. With the formalization in terms of symmetric and anti-symmetric state functions, these issues could no longer be avoided. Thus, in 1926, Born, while defending the corpuscular as opposed to wave-like conception, acknowledged that these corpuscles could not be identified as individuals.[340] Likewise, Heisenberg noted, in the same year, that Einstein's theory of the ideal quantum gas implies that the

[337] Nordheim 1928, p. 691.

[338] Ornstein and Kramers 1927.

[339] Further work was continued along these lines by Halpern and Doermann and others; see ter Haar 1955.

[340] Born 1926.

"individuality of the corpuscle is lost".[341] Shortly after, Weyl, in his typically evocative fashion, wrote that,

> … the possibility that one of the identical twins Mike and Ike is in the quantum state $E_1$ and the other in the quantum state $E_2$ does not include two differentiable cases which are permuted on permuting Mike and Ike; it is impossible for either of these individuals to retain his identity so that one of them will always be able to say 'I'm Mike' and the other 'I'm Ike.' Even in principle one cannot demand an alibi of an electron![342]

This understanding of what the new quantum statistics apparently implied—that quantum particles have in some sense lost their identity and cannot be regarded as individuals—was so ubiquitous that we shall refer to it as the 'Received View' of particle (non-) individuality.

That this view sits at the heart of attempts to come to grips with the new physics can be seen if one takes a look at the proceedings of the Solvay Conference of 1927 at which the 'Copenhagen Interpretation' effectively came to be cemented into place.[343] There we find Langevin, for example, commenting on the difference between the 'old' and 'new' statistics.[344] With regard to the former, he highlights the essential role played by the supposition of the individuality of the constituents of the system,[345] but then notes that, in the new statistics, this individuality of the constituent particles must be abandoned, to be substituted by the individuality of the 'states of movement'. If whatever number of 'constituents' are allowed to occupy a single such state one obtains Bose-Einstein statistics and if only one particle can occupy a state, Fermi-Dirac statistics results.[346] A further distinction can then be established between photons and material particles in that the latter, but not the former, are 'impenetrable'.[347] In either case, however, what this new 'mode of representation' implies is that by virtue of being completely identical (or indistinguishable in our terms), individuality cannot be attributed to the particles themselves, but only to the states of motion.[348] This point, that the loss of individuality of the particles was accompanied by an attribution of individuality to the states, was also emphasized by Schrödinger, as we shall see.

Consideration of these issues is not restricted to commentaries on the nature of the new physics but also arises in the context of its applications. One of the most significant was Heitler and London's analysis of the hydrogen, or

[341] See Miller 1987, p. 310.

[342] Weyl 1931; this is taken from the English translation of his book but it can also be found in the German original from 1928.

[343] See Cushing 1994.

[344] Solvay 1928, pp. 268–9.

[345] Similarly, Lorentz emphasizes the individuality of the electron in his preceding discussion of determinism (Solvay 1928, pp. 248–9).

[346] Ibid., p. 269.

[347] Ibid.

[348] Ibid.

homopolar, bond in terms of anti-symmetric wave functions.[349] It was clear that the attraction between two hydrogen atoms could not be accounted for in terms of Coulomb forces. The solution to the problem lay with the non-classical exchange integral, introduced by Heisenberg in the work recorded above. The understanding of this concept underwent a shift from the idea of a literal exchange of electrons to the perception that its basis lies in the non-classical indistinguishability of the electrons.[350] Heitler and London's paper is situated right at the heart of contemporary struggles to grasp the essentially metaphysical implications of the new quantum mechanics. On the one hand, the mathematics employed harkened back to the literal picture of exchange and the idea of beats; on the other, they noted that the electrons could not be regarded as labelled. On this basis the electronic wave function of the two-atom system could be written in either symmetric or anti-symmetric form. With the electron spins incorporated, the Pauli Exclusion Principle dictates that the anti-symmetric form be chosen, with spins anti-parallel. This corresponds to the state of lower energy and hence attraction. With the introduction of spin and the Exclusion Principle chemical valence and saturation could be understood and the 'problem of chemistry' solved, leading Heitler to exclaim, now "[w]e can … eat Chemistry with a spoon".[351] In a subsequent letter to Born, London wrote, " … we were very proud when we realized that we get the exchange degeneracy because of the similarity of the electrons".[352]

Several years later, in his analysis of what has come to be known as the 'measurement problem', co-authored with Bauer and written with the intention of rendering von Neumann's classic work accessible to physicists,[353] London insisted that although we can attribute pure states to those particles which have been prepared using a suitable measurement arrangement, we cannot know which atoms have the property in question since the preparation never puts any individual object into a new pure state. Thus, our attribution of the pure state is bought at the cost of the individuality of the object, which remains absolutely 'anonymous'.[354] As we shall see in subsequent chapters, this notion of the anonymity of quantum particles crops up again in recent philosophical discussions of individuality in quantum mechanics. According to London and Bauer, this anonymity does not conflict with physical practice because most

[349] A detailed discussion can be found in Gavroglu 1995, Ch. 2.

[350] Carson 1996.

[351] Gavroglu op. cit., p. 54.

[352] Ibid., pp. 87–8.

[353] For a discussion both of the importance of the London and Bauer monograph for the post-war appreciation of the nature of the measurement problem, and London's phenomenological (as in Husserlian) leanings, see French 2002.

[354] London and Bauer 1983, p. 257.

measurements are not to do with the properties of individual systems but rather with general properties of species of systems. They conclude:

> Quantum mechanics, truly a 'theory of species', is perfectly adapted to this experimental task. But given that every measurement contains a macroscopic process, unique and separate, we can hardly escape asking ourselves to what extent and within what limits the everyday concept of an individual object is still recognizable in quantum mechanics.[355]

This latter question was addressed by Bohr and we can see in these remarks by London and Bauer elements of Bohr's philosophy of 'complementarity' as it relates specifically to issues regarding individuality. These issues tend to be overlooked in discussions of Bohr's view and given his importance we shall devote a few pages to spelling out the details.

## 3.4 BOHR'S VIEW OF PARTICLE INDIVIDUALITY

Bohr first mentioned his doctrine of 'complementarity' in an early draft of his famous talk at Como in September 1927.[356] It has become common to understand this doctrine more or less straightforwardly in terms of wave-particle duality. However, this is not how it is presented in the paper itself.[357] Rather, complementarity is expressed in terms of the contrast between spatio-temporal and causal descriptions, which together characterize classical physics: insofar as one may describe a physical phenomenon in terms of the former, utilizing position, for example, one is precluded from describing it in terms of the latter, in terms of momentum. Furthermore, and significantly, Bohr cashed it out as a contrast between the superposition principle of wave mechanics and the 'assumption of the individuality of particles'[358], this contrast reflecting

[355] Ibid.

[356] Kalckar 1985, pp. 26–7. There has been considerable discussion regarding the philosophical origins of complementarity (see, e.g., Jammer 1966, pp. 172–9); however, Kalckar emphasizes its development as a response to problems in the foundations of quantum physics. According to Beller, the central message of this paper was the announcement of a 'happy marriage' between Schrödinger's wave-theoretical treatment and Bohr's fundamental quantum postulate, as expressed through the notion of discrete atomic 'stationary states' (Beller 1992, p. 149).

[357] According to Beller, it was the issue of interactions, between particles, between particles and atoms, between radiation and atoms, that was the centre of attention at the time, rather than wave-particle duality (op. cit.).

[358] Bohr 1985, p. 135 and p. 157. (The first page reference corresponds to the text of the Como talk published in the proceedings of the conference in 1928; the second to that of Bohr's 1928 paper to *Nature* which reproduces in large part but also extends the content of the Como talk; see Kalckar op. cit.). That Bohr expressed complementarity this way has, of course, been noted before. Thus Beller writes that, "… for Bohr, uncertainty followed from a dialectical combination of continuity and discontinuity, or individuality and superposition" (Beller op. cit., p. 174). And Murdoch notes that Bohr "… frequently speaks of 'individuality and superposition' rather than of 'particle' and 'wave' " (Murdoch 1987, p. 56). However, in neither case is this invocation of individuality fully explicated, nor is it related to the implications of quantum statistics.

exclusive but complementary aspects of the situation.[359] Let us consider what he meant by this assumption, in particular.

Bohr begins the substantive part of his paper (the version which was published in the proceedings of the conference) with a characterization of 'individuality' in terms of the 'essential discontinuity' of quantum mechanics:

> Notwithstanding the difficulties which hence are involved in the formulation of the quantum theory, it seems, as we shall see, that its essence may be expressed in the so-called quantum postulate, which to any atomic process attributes an essential discontinuity or rather individuality, completely foreign to the classical theories and symbolised by Planck's quantum of action.[360]

Bohr understood this postulate in terms of the unavoidable disturbance of atomic phenomena by an act of observation. This leads directly to the complementarity of space-time and causal descriptions, since to define the state of the system requires the absence of disturbance, but then "any possibility of observation is excluded".[361] If observation is allowed, then the interactions involved prevent us from giving an unambiguous definition of the state and "there can be no question of causality in the ordinary sense of the word".[362] Thus Bohr regarded his doctrine of complementarity as following from the quantum postulate which embodies the 'essential' individuality in the above sense of a fundamental discontinuity. However, there is more to his account than just this and it is more directly relevant to the theme of the present chapter.

Bohr continues by exploring this view in the context of the quantum analysis of the nature of light and material particles. Significantly, with regard to the latter he emphasizes that "[t]he individuality of the elementary electrical corpuscles is forced upon us by general evidence".[363] What he is referring

[359] It is surely significant that this is how complementarity is characterized in one of the first books to be published in the aftermath of the development of the new quantum mechanics: "The quantum theory replaces the causal spatio-temporal coordination of classical physics by two apparently contradictory but actually complementary ideas of individuality and superposition" (Birtwhistle 1928).

[360] Bohr 1985 p. 114. The same phrase can be found in drafts dated to September and October 1927 and also in his 1928 paper in *Nature*; see Bohr op. cit., pp. 75, 91 and 148.

[361] Ibid., pp. 114–15.

[362] Ibid., p. 115.

[363] Ibid., p. 116. Again, this can be found in both the earlier drafts and the later *Nature* piece; ibid., pp. 77, 92 and 149.

to here is the sort of thing one sees on a scintillation screen for example: discrete flashes of light that lead one to suppose—given the classical metaphysics within which one is operating—that what is causing those flashes are individual particles, where individuality is to be understood in terms of Space-Time Individuality, as what the flashes indicate is the spatio-temporal location of the particles. Again, we shall return to this sense of evidentially based individuality in subsequent chapters. Nevertheless, Bohr acknowledges, there is also evidence—concerning the reflection of electrons, for example—which requires the use of " … the wave theoretical superposition principle in accordance with the original idea of L. de Broglie".[364] Hence the evidence presents us with a dilemma, which is resolved by accepting that what we are dealing with here are 'complementary pictures' of the phenomena. Furthermore,

> … it must be kept in mind, that according to the view taken above, radiation in free space as well as isolated material particles are abstractions, their properties on the quantum theory being observable and definable only through their interactions with other systems. Nevertheless these abstractions are … indispensable for a description of experience in connexion with our ordinary space-time view.[365]

These twin themes—of the complementarity between superpositon and individuality and the nature of both as essential abstractions—are returned to again and again in this work.[366]

The final passage of this paper is particularly interesting with regard to our history of quantum statistics. Here Bohr records the introduction of the spin, or 'magnetic moment', of the electron which, together with Heisenberg's 'resonance' analysis of multi-electron atoms, have 'brought to completion' an understanding of both atomic spectra and the periodic table.[367] However, he notes, the 'intimate connection' between the Pauli Exclusion Principle

[364] Ibid., p. 116.

[365] Ibid.

[366] In an unpublished manuscript dated 13 September 1927, Bohr writes, "The complementary features of the apparent contrary claims of individuality and superposition finds its explanation thereby that such agencies as free material particles and radiation in empty space are abstractions according to the quantum theory. They can only be observed through their interactions' (Ibid., ll. 4 –5). Later on in this manuscript he continues, somewhat cryptically: "Heisenberg's conclusions but use that it rests on the discontinuity character of process only by a necessity of conjugating individuality with Wave superposition" (ibid., p. 80). There is a minor issue as to the transcription of Bohr's handwriting (the text concerned is reproduced on pp. 81–8) as further on still he is recorded as writing "Superposition Schrödinger Heisenberg. Harmony between [Separation?] and [individuality?]" (ibid., p. 80), whereas it seems to us that a good case can be made for 'superposition' rather than 'separation', which meshes better with his articulation of complementarity.

[367] Ibid., p. 136. In the version published in *Nature*, Bohr also records Dennison's application of spin to protons in the nucleus (ibid., p. 158).

and this understanding excludes the hope of an elucidation of the difference between the statistical behaviour of electrons and that of light quanta, that is, of the difference between Fermi-Dirac and Bose-Einstein statistics. This point is repeated in the version of the paper published in *Nature*, but, interestingly, there Bohr refers to " … the 'individuals' symbolised through the conception of light quanta"[368] and also notes that with regard to the 'recent development of statistical theories', the Exclusion Principle is just one among several possibilities.

An even more explicit reference to the new quantum statistics can be found in the published version of Bohr's 1930 address to the Chemical Society upon being awarded the Faraday Medal.[369] Thus he discusses the role of 'the so-called *quantum statistics*'[370] in explaining the spectra of molecules consisting of two 'identical' atoms and notes that for helium molecules, as for light quanta, the wave function must be symmetrical. Nevertheless, there is a crucial difference between photons and helium nuclei in that, in the former case, the statistics reduces in the limit to the classical treatment of electromagnetism, whereas in the latter, " … the new quantum statistics find no unambiguous application within the scope of ordinary statistical mechanics in which the existence of the action quantum is neglected and the particles are treated as *individual* dynamical entities".[371] Again, as with quantum notions in general, these new statistics have an "essentially non-visualisable character".[372]

This non-classical aspect of the new statistics is then explicitly related to issues of identity in the context of the problem of β-decay: straightforward application of conservation of energy implies a variation in the mass of nuclei corresponding to the same radioelement.[373] Yet such a variation is ruled

[368] Bohr 1985, p. 158.

[369] Ibid., pp. 371–408.

[370] Ibid., p. 397; his emphasis. Bohr had earlier noted the connection between the statistics and the symmetry of the wave function (ibid., pp. 394–5). With regard to the Exclusion Principle, he writes that it can be formally represented by an anti-symmetrical wave function "… *although* all electrons are indistinguishable from one another and in consequence play equivalent parts in the wave function" (ibid., p. 394; our emphasis).

[371] Ibid., p. 398; our emphasis. He also notes the difference from the classical situation in that there exist two forms of quantum statistics. With regard to classical statistics, Bohr was rather critical of Boltzmann's work and much preferred the ensemble approach of Gibbs (ibid., pp. 319–26). In the context of his understanding of the canonical ensemble, Heisenberg famously recalled that Bohr emphasized the 'complementarity' between energy and temperature, in the sense that as soon as one knows the temperature, the concept of energy has no meaning (ibid., p. 325).

[372] Ibid.

[373] This problem was, of course, subsequently solved by Pauli's introduction of the neutrino, although prior to this Bohr famously entertained the possibility of a violation of conservation of energy (ibid., p. 407). The point is, he considered quantum statistics to be so well established by this stage as to possess sufficient force to contribute to the toppling of a fundamental conservation principle.

out on the grounds, among others, that the nuclei obey quantum statistics, which means that nuclei of the same kind (possessing the same charge and mass) " … are not to be regarded as approximately equal, but as essentially identical".[374] Furthermore, Bohr writes, "[t]his conclusion is the more important for our argument, because, in the absence of any theory of the intra-nuclear electrons,[375] the identity under consideration is in no way a consequence of quantum mechanics, like the identity of the extra-nuclear electronic configurations of all atoms of an element in a given stationary state, but represents a *new fundamental feature of atomic stability*".[376]

How does this discussion of particle identity relate to Bohr's consideration of the notion of individuality in his presentation of complementarity? The latter notion is understood, at its most fundamental level, in terms of both *discontinuity* and *unity*, which hold for both processes and particles.[377] As applied to the former, this individuality is represented by the quantum of action which gives rise to a profoundly non-classical sense of discontinuity and indivisibility in the sense that every change of state of an atom, for example, should be regarded as an individual process which cannot be subjected to a more detailed description.[378] It was this sense of individuality which Bohr took up in subsequent discussions and carried into the debate with Einstein.[379] As applied to particles, it was to be understood likewise in terms of discontinuity, in the sense that a particle can be contrasted with a wave, and a fundamental kind of unity. It is interesting that with regard to the latter, Bohr draws an

[374] Ibid., p. 406.

[375] Earlier, Bohr notes the following problem: both helium, with an even total number of protons and 'intra-nuclear' electrons, and nitrogen, with an overall odd number of particles but an even number of protons, satisfied Bose-Einstein statistics. Indeed, experimental evidence suggested that nuclei with an even/odd number of protons obeyed Bose-Einstein/Fermi-Dirac statistics respectively. But this implied a peculiar 'passivity' of the intra-nuclear electrons when it came to the statistics. Bohr saw this as a "… very direct indication, indeed, of the essential limitation of the idea of separate dynamical entities when applied to electrons" (ibid., p. 404). Even more interestingly, from the perspective of our discussions of quasi-set theory in later chapters, Bohr continues, "[s]trictly speaking, we are not even justified in saying that a nucleus contains a definite number of electrons, but only that its negative electrification is equal to a whole number of elementary units …" (ibid.). The concept of 'intra-nuclear' electrons was eventually abandoned with the advent of Fermi's theory of β-decay, which Bohr had accepted by 1936 (Bohr 1936, pp. 25–6; 1985, Vol. 5, pp. 215–16).

[376] Ibid.

[377] Murdoch op. cit., p. 56.

[378] See, for example, Bohr 1929b.

[379] Thus in his response to the EPR paper, Bohr wrote, "The impossibility of a closer analysis of the reactions between the particle and the measuring instrument is indeed no peculiarity of the experimental procedure described, but is rather an essential property of any arrangement suited to the study of the phenomenon of the type concerned, where we have to do with a feature of *individuality* completely foreign to classical physics" (Bohr op. cit., Vol. 5, p. 293); see also Folse 1985, pp. 150–1.

analogy with the individuality of persons:

> … the apparent contrast between the continual onward flow of associative thinking and the preservation of the unity of the personality exhibits a suggestive analogy with the relation between the wave description of the motions of material particles, governed by the superposition principle, and their indestructible individuality.[380]

This understanding of individuality allowed Bohr to talk of 'individual' light quanta, even though such quanta obey non-classical statistics, as we have seen. It is also worth noting his emphasis that this form of individuality is indestructible. In the 'introductory survey' to the above volume he notes the fundamental dilemma presented by the experimental confirmation of de Broglie's wave theory: that although matter may exhibit wave-like characteristics, "… there can be no question of giving up the idea of individuality of the elementary particles; for this individuality forms the secure foundation on which the whole recent development of the atomic theory depends".[381] Bohr continues by insisting that, "[t]he main purpose of the article [namely the Como paper] is to show that this feature of complementarity is essential for a consistent interpretation of the quantum-theoretical methods".[382]

What the sense of individuality as applied to processes effectively did was to cleave apart the two most notable descriptive characteristics of individual particles, whether they be light quanta or electrons in atoms. These have to do with acting as a carrier of energy/momentum and following a distinct spatio-temporal trajectory, respectively. It is in terms of these two characteristics—with the former understood as underpinning a causal description—that complementarity is originally formulated, as we have seen. However, Bohr effectively privileges the standard understanding of particle individuality in terms of a spatio-temporal description—that is, what we have called Space-Time Individuality—and thus this sense of complementarity is also spelled out as that which holds between superpositions and individuality, taken in *these* terms. The former allows us to understand the notion of stationary state and in particular its 'supramechanical' stability but its invocation precludes any spatio-temporal description and, in particular, any specification of the behaviour of the separate particles in the atom. Thus the two applications of individuality work together here: as applied to processes it underpins the very notion of a stationary state, but this precludes its application to particles.

[380] Bohr 1929, pp. 99–100.

[381] Bohr 1934, pp. 10–11.

[382] Ibid.

This approach allows an 'unambiguous definition' of the energy of the atom and the 'fundamental renunciation' of the space-time description is unavoidable when it comes to the interpretation of observations. On the other hand, in order to 'unambiguously use observations regarding the behaviour of the particles in the atom' one must neglect interactions and treat the particles as free and thus as distinct individuals, but then one cannot invoke stationary states or say anything about the overall energy. Likewise, outside the atom, the individuality of the particles seems forced upon us by the experiments but we recall that what these experiments register—by the (discontinuous) flash on the scintillation screen for example—is the spatio-temporal location of the particle.[383] Thus they preclude any consideration of the wave-like nature through superpositions. An account of the latter is complementary to that of (spatio-temporal) individuals and both superpositions and individuals are to be regarded as abstractions upon which our complementary descriptions rest.[384]

Can we understand the 'superposition' half of the complementarity relationship as a kind of non-individuality? Certainly, if the nature of complementary relationships is cashed out in terms of mutually exclusive or even inconsistent notions, one would have to see superposition as somehow opposed to individuality. And, as we shall see, this is indeed how superposition has been regarded, if it, or more generally quantum 'entanglement', is understood as denying the separability of states which, it is alleged, Einstein, in his later debate with Bohr, took to be definitive of individuality. Can this idea of superposition-as-a-loss-of-individuality then be related to the acknowledged implications of quantum statistics? Bohr seems to have thought so: in a 1928 letter to Bohr, Schrödinger argued that, if Heisenberg's uncertainty principle is right, then the notion that there are discrete quantum states cannot be experimentally tested:

> One may … verify this fact in a few simple cases, e.g., in the quantization of an ideal gas. If we allow the molecule a latitude in position of the size of the entire gas volume, then the uncertainty in the momentum becomes of the order of magnitude of the momentum difference of neighbouring quantum states.[385]

To this Bohr replied by invoking the complementarity between spatio-temporal and causal descriptions where this is revealed by the 'apparent

[383] It is on this point that the whole Bohr interpretation ultimately rests.

[384] Folse understands them to be abstractions in the sense of 'purely theoretical' representations, as opposed to descriptions of the properties of independent, real objects existing separately from observations. The latter properties correspond to those of visualizable pictures, whereas the former do not (op. cit., p. 117).

[385] Ibid., p. 47; Schrödinger's emphasis.

contrast' between the superposition principle and the individuality postulate.[386] A 'particularly striking' example is the 'absolute exclusion' between the application of the notion of stationary states and the spatio-temporal 'tracking' of the individual particles in the atom.[387] He writes:

> In the article [his 1928 *Nature* paper] I have endeavoured strongly to stress the failure of classical pictures in the quantum theoretical treatment of the interaction problem, and to emphasize that our entire mode of visualization is based on the abstraction of free individuals—a point where, in my opinion, the relationship between classical theory and quantum theory is particularly evident. I might add further that just in the case, touched upon in your letter, of the quantization of a gas, this failure shows up so strikingly in the paradoxes of the new statistics.[388]

And he concludes by rejecting Schrödinger's use of the uncertainty principle in this case, "because here of course the momentum variable conjugate to the coordinate does not have an unambiguous value".[389]

In a later letter to Pauli, in 1929, he writes,

> Thus one can pursue the reciprocity of the concept of individuality and the superposition principle to far-reaching consequences. One can show in general that any use of the former concept limits the application of the latter principle as an immediate consequence of the loss of phase resulting from every observation. Conversely, any consistent application of the superposition principle limits the possibility of a visualizable interpretation based on the principle of individuality, as it finds expression above all in the quantum theoretical treatment of systems with several identical particles. All this contains of course nothing really new.[390]

In other words, the paradoxical implications of the new statistics have to do with the loss of a 'visualizable interpretation based on the principle of individuality', something to which Bohr subsequently refers in his later comments on

[386] Bohr 1934, p. 48.

[387] Ibid., p. 49.

[388] Ibid., pp. 49–50. It is also worth noting here Schrödinger's suggestion that the new quantum theory requires 'new concepts' (op. cit., p. 47), whereas Bohr insists that not only can he see no basis for any such new conceptualization but "… the 'old' empirical concepts appear to me to be inseparably linked to the foundations of the human means of visualization" (ibid., p. 48). We might speculate on a possible connection between Schrödinger's suggestion here and his subsequent willingness to abandon identity and individuality for particles. For Bohr, of course, we cannot give up individuality entirely as it is both empirically grounded and necessary for 'visualization'.

[389] Ibid., p. 50. In a letter to Einstein, Schrödinger insists that he does not understand at all what Bohr is on about here (Bohr op. cit., p. 50). Einstein, who had been given copies, by Schrödinger, of both Schrödinger's earlier letter and Bohr's response, agrees and it is here that he makes his famous remark that "[t]he Heisenberg-Bohr soothing-philosophy—or religion?—is so cleverly concocted that for the present it offers the believers a soft pillow of repose from which they are not so easily chased away. Let us therefore let them rest" (ibid., p. 51).

[390] Ibid., p. 193 and p. 443.

these statistics.[391] Bohr insisted that one cannot rest content, as it were, with this loss of individuality, since the very foundation of a space-time description is provided by the 'abstraction' of free individuals[392] *and* this abstraction is pressed upon us by the experimental circumstances indicated above. What the loss represents is only one *side* of the complementary relationship, with individuality providing the other. In effect, then, Bohr encapsulated, within his complementarity interpretation, the two metaphysical packages that we will be exploring in this book.

## 3.5 BORN AND STRUCTURAL INVARIANTS

Nevertheless, by the early 1930s the 'Received View' of quantum particles as non-individuals had propagated from specialist to more popular texts: Darwin, in his 'semi-popular' presentation of the new mechanics understands the Exclusion Principle as insisting that "… we ought not to talk about electrons as having any individuality".[393] He notes, however, that as it stood then, the Principle was expressed in a manner that would probably come to be seen as rather clumsy: " The description consists in starting as though the electrons had full individuality, and then showing that this individuality did not matter".[394] The idea is familiar: we first assume the electrons are individuals and can be labelled as such in constructing the wave function for the assembly and we then symmetrize *à la* Schrödinger, effectively denying the individuality by imposing the anti-symmetry condition. Interestingly, Darwin then offers an alternative 'rough and ready' account which avoids this clumsiness by denying the individuality of the electrons from the very start. At the heart of this account is the idea that the Exclusion Principle represents the quantum replacement of the venerable Impenetrability Assumption (as discussed in Chapter 1), with the twist that since the relevant wave functions can overlap, two electrons can be in the same place at the same time, as long as they are not "doing the same thing".[395] If they are, then there is a kind of repulsion and the Coulomb force between electrons comes to be seen as an aspect of

[391] Miller writes that,"… for Bohr and others, not only had the bound electron lost its visualizability in the classical meaning of this term, but free electrons had lost their individuality" (Miller 1987, p. 308). As we hope to have indicated, this 'loss' is not absolute for Bohr, but represents only one complementary aspect of the situation.

[392] op. cit., Vol. 6, p. 131.

[393] Darwin 1931, p. 168.

[394] Ibid., p. 169.

[395] Ibid., p. 179.

Exclusion.[396] This suggestion of a connection between the Exclusion Principle and Coulomb repulsion is only the first step, however, and the goal is to arrive at a theory in which the electron, as a particle, emerges, in some sense, as a result of observation:

> As far as concerns the wave aspect there will cease to be electrons at all, but only a sort of electrical juice. It will be the step from wave to particle which automatically cuts the juice into units, and thus it will be the observation that creates the electron. In some such way as this, the electrons would have no individuality from the very start, which is what we would like.[397]

It is hard to resist the temptation to read into these remarks something like an early form of the quantum field theoretic view of particles.

We have already noted Born's advocacy of the Received View and in his 1943 defence of the role of experiment in physics,[398] he not only argues, again, that quantum particles lack individuality but emphasizes the experimental basis of this conclusion. Thus, in his outline of the development of statistical mechanics, after noting that the description of the relevant statistical weights is simpler for quantum systems than for classical ones in that "… each state of given energy which by no physical means can be split into several states has the same weight",[399] he insists that if photons are treated as "… having an individuality of their own",[400] we would not obtain Planck's (experimental) law for black-body radiation.[401] Instead, he continues, we must assume "… that two states which differ only by the exchange of two photons are physically indistinguishable and have statistically to be counted only as one state. In other words, photons have no individuality".[402]

As extended to particles by Bose and Einstein, this assumption would have remained a 'theoretical speculation' because of the experimental difficulties in attaining the requisite conditions. However, Pauli's exclusion principle gives much more direct evidence for the "lack of individuality of particles",[403] resting as it does on such 'facts of observation'[404] as the non-existence of the lowest state of the helium atom as well as 'innumerable consequences' such as, and most importantly, Bohr's account of the periodic table.

[396] As Darwin acknowledges, this curious idea of unifying the Coulomb force and quantum mechanical 'Exclusion' was due to Eddington, whose views of quantum indistinguishability are discussed in French 2003.

[397] Ibid., p. 182.

[398] Born 1943.

[399] Ibid., p. 27.

[400] Ibid.

[401] We have already seen how problematic such a claim is.

[402] Ibid., pp. 27–8.

[403] Ibid., p. 29.

[404] Ibid.

Now, Born's defence of the Received View of particle (non-) individuality is particularly significant because it is to Born, of course, that the orthodox, Copenhagen, interpretation owes its understanding of the wave function: namely as giving the probability of obtaining a given result on measurement, such as the position of an electron, say. The ontology here is that of particles and was conceived of and presented as the alternative to Schrödinger's problematic wave-based world-view. But now a tension arises within this orthodox view: if an electron, for example, can no longer be regarded as an individual, how can this particle ontology be maintained? As we shall see, Schrödinger took this tension to undermine the orthodox interpretation and invoked non-individuality to support the later development of his own view. Born himself suggested a way of removing the tension, which, although rather tentative, does bear some similarity to one aspect of Schrödinger's later interpretation.

The broadly philosophical context of Born's ideas is made explicit in subsequent reflections, delivered as a talk in 1958.[405] Again Born states quite bluntly that "particles have no individuality"[406] and in the very next line repeats the point about the non-classical counting. Here, however, he raises the question how we are to talk of electrons, say, as particles of a certain kind, given this lack of individuality. His answer has two features to it. The first is straightforward and involves the 'widening' of concepts, in the sense that the concept of 'number' for example, has been extended to include transcendental and imaginary numbers. The justification for this extension lies in the fact that some subset of relevant properties is shared. And so, "[w]e call an electron an elementary 'particle' because it has many, even if not all, of the properties particles exhibit in the tangible world".[407]

The second feature is more significant and concerns the ontological nature of particles, as things which are not individuals yet can be said to 'have' properties, and here an interesting structuralist element emerges. He begins with an analysis of the trans-temporal identity of perceptible objects, such as a bird or a dog, and drawing on both Gestalt psychology and then current research in the physiology of perception, argues that what we are 'given' are not separate sense impressions—as the positivists believed—but rather 'certain invariant characteristics' of these impressions, which we then describe as 'objects'.[408] Moving along the epistemological spectrum to the unobservable, we again

[405] Born 1962, pp. 14–37.

[406] Ibid., p. 31.

[407] Ibid., p. 25.

[408] Ibid., p. 26.

seek for 'invariant traits' but now they must be revealed through mathematical analysis:[409]

> The properties of the postulated physical reality, which are independent of the reference system, are the invariants which can be obtained by the analysis of coordinates or projections.[410]

Reality is 'projected' via experiments, in general, and in the quantum context, by complementary experiments in particular. By making such complementary experiments we can obtain 'sets of invariant magnitude' which allows us to talk of electrons, protons etc. as particles of a certain kind. Thus objectivity in quantum physics is grounded not in an ontology of individual objects but in one of structural invariants.[411]

It is not difficult to find other expressions of the Received View during this period, particular in popular expositions where physicists felt they could 'cut loose' philosophically, or historically. Oppenheimer, for instance, noted how the modern notion of 'elementary particle' deviates from earlier understandings with respect to both elementarity and identity:

> We are continuing the attempt to discover, to identify and characterize, and surely ultimately to order, our knowledge of what the elementary particles of physics really are. I need hardly say that in the course of this we are learning again how far our notion of elementarity, of what makes a particle elementary, is from the early atomic ideas of the Hindu and Greek atomists, or even from the chemical atomists of a century ago. We are finding out that what we are forced to call elementary particles retain neither permanence nor identity, and they are elementary only in the sense that their properties cannot be understood by breaking them down into subcomponents.[412]

[409] Here Born cites Klein (ibid., p. 28), which is somewhat ironic given his earlier opposition to group theory: in a 1930 letter to Fritz London (an ardent advocate of the role of group theory in physics) Born dismissed it as "… not in accord with the way things are" (Gavroglu 1995, p. 56).

[410] Ibid., p. 28.

[411] Born was not the only one to respond to the apparent loss of individuality of quantum particles by shifting to a structuralist ontology, as similar moves were made by Cassirer and Eddington (see French 2003). He goes on to respond to a criticism of this structuralist approach by a Soviet physicist Suvorov who insisted that "the idea of invariance does not answer the fundamental epistemological question 'What is reality?' " (ibid., p. 31). Suvorov argued that focusing on certain invariant properties such as charge and mass was not sufficient for the identification of certain kinds of particles, since, for example, the anti-proton has the same mass as the proton and the same charge as the negative hydrogen ion. Therefore, certain 'relationships' which are 'specific' to a particular kind of object must be sought for and this restores an object-based ontology. Born's response is succinct and obvious: these 'specific relationships' can be taken to designate the totality of invariant properties associated with a particle. There is a nice parallel here with recent discussions about object vs. structural ontologies in the context of structural realism (see French 2001; French and Ladyman 2002).

[412] Oppenheimer 1947, p. 200.

## 3.6 SCHRÖDINGER AND THE LOSS OF IDENTITY

The most well-known articulation of the Received View can be—and typically has been—attributed to Schrödinger. We have already noted his concern with particle identifiability and individuality in his consideration of the *N*! problem. Indeed, this concern fed directly into the construction of his wave mechanics. Following the development of quantum statistics within the framework of the new mechanics, Schrödinger—one of the more philosophically inclined of the quantum physicists[413]—returned to his earlier concern and on numerous occasions expressed the claim that particles could no longer be regarded as individuals. However, and significantly, he also took very seriously—more so than most physicists—the implications of this claim for the interpretation of quantum mechanics. We have already noted the tension that arises within the orthodox interpretation if particles are regarded as non-individuals and Schrödinger argued that this tension was so great, it actually undermined the orthodox view itself. What was required, he insisted, was a new interpretation capable of accommodating this non-individuality.

Perhaps the most well-known statements of the Received View appear in a series of public lectures[414] given by Schrödinger at the Institute for Advanced Studies in Dublin in 1950, subsequently published as *Science and Humanism*.[415] He writes that, in the face of quantum physics,

> … we have … been compelled to dismiss the idea that … a particle is an individual entity which retains its 'sameness' forever. Quite the contrary, we are now obliged to assert that the ultimate constituents of matter have no 'sameness' at all.[416]

And he continues,

> I beg to emphasize this and I beg you to believe it: It is not a question of our being able to ascertain the identity in some instances and not being able to do so in others. It is beyond doubt that the question of 'sameness', of identity, really and truly has no meaning.[417]

[413] See Bitbol and Darrigol (eds.) 1992; Bitbol 1996a.

[414] So again they feature in what were primarily popular expositions.

[415] Schrödinger 1952; 1996.

[416] Schrödinger 1996, p. 121.

[417] Ibid., pp. 121–2. He refers to the particles' 'lack of individuality' in several places (see, for example 1996, p. 151) and bluntly states, "Atoms—our modern atoms, the ultimate particles—must no longer be regarded as identifiable individuals" (1996, p. 162).

We shall return to the issue of how one might represent and understand this idea of identity no longer having any meaning in subsequent chapters,[418] but for the moment we simply want to consider, first of all, the grounds for such claims, and secondly, what Schrödinger took their implications to be.

These claims were based on two sets of considerations.[419] The first has to do with the re-identifiability or trans-temporal identity of particles. We recall from Chapter 1 the requirement of spatio-temporal continuity as a necessary condition of trans-temporal identity. Schrödinger's concern here is precisely that this condition fails in the quantum context and thus, significantly, that the very idea of individual particles must be abandoned. This concern can be traced back to his unhappiness with the claim that particles possess values of every observable and his own formulation of no-hidden-variables arguments.[420] When the observables are position and momentum, this sense of metaphysical disquiet led him to conclude first, that particles could not be attributed definite spatio-temporal trajectories and further, that the idea of particles which do not have such trajectories was simply incoherent. As he put it in a letter to Margenau, "To me, giving up the path seems giving up the particle".[421]

According to Bitbol, this step, from giving up trajectories, to giving up particles, is grounded in Schrödinger's view of spatio-temporal continuity as the Principle of Individuality itself.[422] As Schrödinger explicitly acknowledged, this idea of individual particles travelling along well-defined trajectories can be traced back to the pre-Socratic atomists.[423] In his letter to Margenau,

[418] Moore reports, intriguingly, that in the summer of 1948, Schrödinger visited Russell in Portmeirion and discussed his view of the loss of particle identity with the great philosopher (op. cit., p. 443). Unfortunately there do not appear to be any records of what Russell thought of this view (we have asked the Russell Society but they have not deigned to respond). In his essay 'What is Real', written in 1960, Schrödinger quotes with approval Russell's warning of the possibility of philosophical prejudices arising from the subject-predicate sentence structure of most languages (1964, p. 76). Such 'prejudices' include, according to Schrödinger, the 'obstinately lingering doctrine' of substance and accident and, he continues, we can also attribute to this sentence structure "… the giving of a new lease of life to the identitas indiscernibilium, regarded as covering something which in fact lies at a much deeper level, namely Pauli's principle of exclusion, which, incidentally, should be regarded as an approximation to a more general proposition laid down by Dirac" (ibid.). This is one of the very few references to physics in the essay and Schrödinger is obviously referring to the anti-symmetrization of the wave function in his closing remark.

[419] In his otherwise useful discussion, Bitbol, for example, only considers the first (Bitbol 1995).

[420] See Bitbol 1995, p. 7.

[421] Ibid., p. 8.

[422] Ibid.; see Schrödinger 1996, p. 130.

[423] Schrödinger saw atomism as a response to the 'intricacies' of the continuum represented by the difficulties in accounting for the observable changes in matter using a plenum-based ontology, such as that proposed by Anaximenes (1996, pp. 157–62). His 1948 lectures at University College London, subsequently published as *Nature and the Greeks* (1996), sketched the development of Greek thought with the aim—at least in part—of illuminating the 'present crisis of the fundamental sciences' (ibid., p. 15). And one aspect of this crisis has precisely to do with the loss of individuality of elementary particles. Here Schrödinger's claims seem slightly more qualified than in his later popular expositions. He writes, for example, that "Quantum theory … tells us that what was formerly considered as the most obvious and fundamental property of the corpuscles, namely their being identifiable individuals, has only a limited significance" (1996, p. 16). It is only when a particle is moving sufficiently fast and the density of other particles is sufficiently low (it is in such situations that Maxwell-Boltzmann statistics might be applicable), that its identity "remains (nearly) unambiguous" (ibid.). As we shall see, elsewhere Schrödinger insisted that such thought experiments which involved imagining a particle all on its own were deeply problematic, and that the intrusion of quantum statistical considerations undermined the above possibility of rendering identity 'nearly unambiguous' (see also the remarks by Bitbol, op. cit., pp. 8–9). Indeed, immediately after the above passage, he writes that the issue is not to do with any practical inability to follow a particle but, rather, "… the very notion of absolute identity is believed to be inadmissible" (1996, p. 16).

Schrödinger also insisted that the role of spatio-temporal continuity had been widely recognized within classical physics and could be explicitly found in Boltzmann's work. Boltzmann was, of course, one of Schrödinger's great scientific heroes,[424] and in this context he noted Boltzmann's emphasis on the importance of clear and detailed models[425] which functioned as consistency checks.[426] The construction of such models must meet the demand expressed in the continuity postulate but according to quantum mechanics the latter is unfillable, and this "… is intimately connected with what I have earlier called the lack of individuality of a particle".[427] In other words, under the impact of the uncertainty relations together with the no-hidden-variables arguments, Space-Time Individuality cannot be maintained in the quantum context.[428]

According to Schrödinger, such considerations alone suffice to strongly question the adequacy of the particle concept,[429] but as he also remarks, other reasons can be given. Thus, in a 1949 seminar he writes that the implications of the 'new' quantum statistics will "… play a great role in our future attitude towards atomism".[430] He continues,

> If you think about those new statistics rationally and soundly, you see that they are *illogical, when applied to the individuals or individual events.*[431]

Schrödinger tries to illustrate what he means using coin tosses or 'casts':[432] if one casts a coin six times, say, classical, Maxwell-Boltzmann counting would

[424] See the various remarks in Moore 1989.

[425] Schrödinger 1996, p. 129.

[426] See also Schrödinger's letter to Eddington, reproduced in Schrödinger 1995, pp. 121–2.

[427] Schrödinger 1996, p. 131.

[428] Again, in his discussion of the two-slit experiment, Schrödinger argues that we must give up the idea of tracing the spatio-temporal history of the particle and the 'gap' this represents in our description of observable events is 'very characteristic of the lack of individuality in the particle (1996, p. 151).

[429] Schrödinger 1995, p. 21.

[430] Ibid., p. 112.

[431] Ibid.; his emphasis.

[432] Ibid., pp. 112–13.

yield the combination of three heads and three tails as having the highest probability. The suggestion that the possibilities of zero, one, two, …, six heads are all equally probable—which is what Bose-Einstein statistics yields—seems 'illogical'. Likewise, it is illogical to assume that heads can only appear once in all six casts or not at all, as Fermi-Dirac statistics dictates. How could we account for such bizarre behaviour of the coins? As Schrödinger suggests, one possibility would be to introduce some form of dependence between the probabilities of each cast, but this would have to be 'most intricate'.[433] Alternatively, he argues, we must reconceive our classical notions of individuality. Thus in the above analogy of coin tosses, the tosses themselves are analogous to the particles and 'heads' and 'tails' correspond to the states (the boxes in our balls and boxes illustration). Now, if we consider just a single toss or cast, Fermi-Dirac statistics seems entirely appropriate, since 'heads' and 'tails' are mutually exclusive. But now what is analogous to the electron is not the cast but the notion 'heads' and "Heads is *not an individual* but a property of an individual, namely of a cast".[434] The corresponding analogy for Bose-Einstein statistics—and this has been reproduced many times in discussions of these matters[435]—would be that of money in a bank account; "But again, *the shillings and pennies in your bank account are not individuals*".[436]

He subsequently stressed the importance of quantum statistics even more explicitly:

> There is a general reason for not regarding the elementary particles—electrons, protons, light-quanta, mesons—as individuals, a reason very well known to everybody. When you are dealing with a system that contains equal particles you must wipe out their individuality, lest you get quite the wrong results.[437]

The appropriate statistics must be invoked, or equivalently, the wave function for the system must be symmetrized or anti-symmetrized appropriately, and, Schrödinger emphasizes, this is the case whether you have a large number of particles—as in a gas containing $10^{20}$ molecules—or only a few—as in the case of the two electrons in the helium atom.[438]

He then makes two further points which will be significant for our discussion of the philosophical implications of the Received View. First of all,

[433] Schrödinger 1995, p. 102.

[434] Ibid., p. 103; Schrödinger's emphasis.

[435] See Hesse 1970; Teller 1995.

[436] Schrödinger op. cit.; his emphasis once again.

[437] Schrödinger 1995, p. 32.

[438] This is a stronger stance than his earlier one, noted above, where he suggested that in the case of low particle density, the identity of the particles would be '(nearly) unambiguous'.

he states that this lack of individuality implies that the particles cannot be regarded as *labelled* in any way:

> This means *much more* than that the particles or corpuscles are all *alike*. It means that you must not even imagine any one of them to be marked 'by a red spot' so that you could recognize it later as *the same*.[439]

Secondly, however, he insists that "… *waves* can easily be marked, by their shape or modulation".[440] Just as the light-waves that hit your eye are the same that were emitted by the sun, or the sound waves that reach your ear are the same as those produced by leather hitting willow on the cricket pitch, so the waves of quantum mechanics *have* to be treated as individuals:

> You know that if you deal with such many-body problems as the He-molecule or a gas of $10^{20}$ molecules, by the method of wave-mechanics, the proper modes have to be regarded as distinguished from one another, they have to be treated as true individuals. You must not apply Fermi-Dirac statistics or Bose-Einstein statistics to them, but ordinary Boltzmann statistics: then you obtain the correct results, the same as you get by applying the new-fangled statistics to the non-individual corpuscles.[441]

Here we see another expression of that form of duality that runs through the history of quantum statistics, where the apparent loss of individuality of the particles is metaphysically compensated for, as it were, by the individuality of the underlying states, or waves. What is the principle of individuality in the latter case? Schrödinger makes perhaps the most obvious suggestion, that it has to do with the 'shape or modulation' of the waves, that is, with their *form*.[442]

Now, these considerations have a two-fold implication for Schrödinger's interpretation of quantum mechanics: the first—very well known—has to do with his focus on a wave-based ontology; the second—less well known—relates to certain structuralist aspects of this ontology which bear a resemblance to Born's speculations. With regard to the first, Schrödinger's critique of the metaphysics of individual particles can be seen as a preparatory basis for his 'ontology of $\psi$-waves'.[443] The second implication emerges from Schrödinger's

[439] op. cit., his emphasis.

[440] Ibid.

[441] Ibid., pp. 32–3.

[442] See Bitbol 1995, p. 12.

[443] Ibid., pp. 9–13. As Bitbol emphasizes, this is not a simple-minded return to his earlier view of reality as wave-like (op. cit., p. 4). Rather what we have is the construction of a Boltzmann-type model or 'Bild' which should not be understood as being in correspondence to the world but is related to the experimental facts via something akin to correspondence rules (see Bitbol 1996). In a set of lectures dated from 1952, Schrödinger writes that "The wave functions are mental material for building analytical pictures of real objects in one's mind and performing *thought experiments* on them; the results of these thought experiments are obtained by ample use of mathematics and are then compared with the results of real experiments" (1995, p. 82).

discussion of the problem of how we obtain individual objects from assemblies of apparently non-individual ones. It is here that the structuralist facet of his view is revealed. He begins by reflecting on our observational knowledge of states of affairs, which he calls 'true observables'.[444] It is these states, he insists, which really constitute matter, as it is meant by the philosopher. He continues,

> The way in which particles constitute matter is thus a very strange and novel one. We must investigate it more closely. Particles, having no individuality, constitute pieces of matter that have. They do it by giving rise to observables. What we usually call the building material is of a fundamentally different nature from what is built up of it. In current quantum mechanics, this different nature is expressed by the twofold set of mathematical entities we use: vectors and tensors, or wave functions and operators.[445]

How, then, is this transition from non-individual particles to individual, "truly observable, pieces of matter" achieved?[446] Schrödinger's answer seems rather weak at first glance: a number of non-individual particles "… coalesce to form a more extended gathering of a build or constitution not copied in the immediate neighbourhood by a number of similar gatherings of the same build".[447] It is this 'coalescence' or 'gathering' which results in observables, or matter 'in the meaning of the philosopher'.

Schrödinger notes two objections which this approach must meet. The first has to do with individuating the pieces of matter and he asks how the individuality of his pocket knife, understood as a "… system of observables referring in the quantum mechanical way to a gathering of particles",[448] would be maintained when faced with other, apparently indistinguishable, pocket knives. As he says, this is trivial. The knife can be marked or scratched or distinguished and hence individuated spatio-temporally.[449] The second objection is more interesting and bears on the recurrent issue of the experimental foundation of the metaphysics of individuality. Schrödinger notes that we sometimes speak of observables referring to a single particle; but if the latter is a non-individual how can the correspondence between observables and individuality be maintained?[450] In experimental practice, however, we never encounter single particles, for reasons already outlined above. Typically such talk arises in the context of certain thought experiments in which similar particles are effectively removed from the immediate 'neighbourhood', a procedure Schrödinger

[444] See his 1995, p. 98.

[445] Ibid.

[446] Ibid.

[447] Ibid., pp. 98–9.

[448] Ibid., p. 99.

[449] Ibid.

[450] In a sense what Schrödinger is doing is reducing individuality to a kind of distinguishability, where the latter is cashed out in terms of these observables.

refers to as "smoothing out of the rest of the world".[451] Then, or so the thought experiment continues, one only has the single particle to form a 'gathering' and effectively produce matter. But, insists Schrödinger, such thought experiments are extremely misleading and encourage us to transfer the metaphysics of individuality from its legitimate domain to circumstances where it cannot be applied:[452]

> These thought experiments are precisely what has given rise to the wrong idea that even a single particle is an identifiable piece of matter, and thus that matter is composed of particles.[453]

His answer to the question, "How does individuality arise at all in objects composed of non-individuals?"[454] is then used to cast light on the deeper metaphysical question, "What are these particles then, if they are not individuals?".[455] The answer to this question is given by describing what is 'permanent' to a particle, despite its 'lack of individuality'. And both the answer and Schrödinger's way of approaching it are remarkably similar to Born's above.

He begins with the example of an iron paperweight in the shape of a Great Dane which accompanied Schrödinger on his various moves, was left behind when he fled Austria and was subsequently recovered and returned into his possession by a friend. He then asks, on what grounds can he be sure it is the same iron Great Dane? The answer is that "[I]t is clearly the peculiar form or *shape* (German: *Gestalt*) that raises the identity beyond doubt, not the material content".[456]

Schrödinger takes this example to be analogous to the case of particles and the above 'coalescence' or 'gathering' is now understood in structural terms,

[451] Ibid., p. 100. Schrödinger emphasizes that this 'smoothing out' leads to difficulties that are conceptual (or, we would suggest, metaphysical) rather than mathematical. Mathematically, as he puts it, one simply constructs the appropriate wave function, suitably symmetrized or anti-symmetrized, for as many particles as one wants to consider (ibid., p. 102). Conceptually, the procedure is deeply problematic and the exclusion of all the other particles is a "much more momentous and possibly dangerous simplification" (ibid.) than it is in the classical case because of this issue of particle (non-) individuality.

[452] The problematic nature of taking a system to be isolated from its environment is also emphasized in Schrödinger's 1937 discussion of Eddington's ideas (Moore op. cit., p. 329). Eddington also maintained that relativistic considerations dictated that one could never physically consider a single particle in isolation.

[453] Ibid., p. 99.

[454] Schrödinger 1996, p. 122.

[455] Ibid.

[456] Ibid., p. 123; his emphasis.

as the individuality of bodies composed of particles, "… arises out of the structure of their composition, out of shape or form, or organization …".[457] In particular, the 'old idea'—outlined in Chapter 1—that individuality is grounded in the identity of substantial matter is now seen as a "… gratuitous and almost mystical addition".[458] It has been replaced by the 'new idea' that "… what is permanent in these ultimate particles or small aggregates is their shape and organization".[459] Since Aristotle, Schrödinger claims, we have been led to believe that 'shape' or 'form' must be the shape or form of something, some underlying substantial substratum which carries individuality, but when it comes to the ultimate particles we must give up this way of thinking. Such particles are, "… *pure shape*, nothing but shape; what turns up again and again in successive observations is this shape, not an individual speck of material".[460]

Of course, by 'shape' or Gestalt here Schrödinger does not mean geometrical shape; the latter features only as part of our crude and childish models.[461] In the more sophisticated models constructed by science, the Gestalt is given by sets of invariant properties and in his 'William James' lectures of 1954, Schrödinger emphasizes the fundamental importance of invariance.[462] Indeed, in a passage which bears a close resemblance to Born's remarks, noted above, Schrödinger states that our basic idea of a 'thing', as something permanent in the world, is the result of this fundamental process of forming invariants and he writes that "[T]he most fundamental invariants that we form at an early stage are the *things* in our surroundings, including our own body".[463]

This process gives us our view of 'what the world is really like', but, he insists, such a view (of permanent things) is not inevitable; indeed, he compares it to 'scientific constructs' and as such, he emphasizes, it is liable to revision and improvement just as scientific theories are. And, of course, it must be revised when we attempt to carry it over from the observable world around us to the quantum domain. In the latter, the invariants give us only the kind (all electrons are fermions, have the same charge, etc.);[464] to get from the kind to the particular thing in the observable, everyday world we must individuate, via space-time trajectories, or substance or whatever, but such individuation must

[457] Schrödinger 1996, p. 124.

[458] Ibid.

[459] Ibid.

[460] Ibid., p. 125; his emphasis. We shall find similar sentiments in Post 1963, for example.

[461] Ibid.

[462] Schrödinger 1995, pp. 141–9.

[463] Ibid., p. 148; his emphasis.

[464] See Bitbol 1995, p. 8.

be abandoned in the quantum realm.[465] Nevertheless, there is permanence, as Schrödinger emphasizes.

That Schrödinger's reflections on the philosophical implications of the 'Received View' should display elements of a structuralist tendency is not so surprising, given his familiarity with the writings of two of the great structuralists of the 1920s and 30s, namely the neo-Kantian philosopher Cassirer and the great scientist and science popularizer, Eddington. Both explicitly accepted the non-individuality of quantum particles and incorporated it within their structuralist metaphysics; however, as interesting as they are, we shall not consider their views here.[466]

We have explored Schrödinger's metaphysics in some detail because of his central importance in articulating and disseminating the 'Received View'. However, it also raises issues of a formal, logico-mathematical nature, as was also perceived in the period following the establishment of quantum mechanics.

## 3.7 WEYL AND THE ANALYSIS OF AGGREGATES

The far-reaching nature of these issues was clearly perceived by Cassirer, and, even more acutely and perceptively perhaps, by Weyl. In remarking on the radical epistemological transformation effected by quantum mechanics, Cassirer notes that "[i]t seemed hitherto to be an axiom, not only of classical physics but of classical logic, that the state of a thing in a given moment is completely determined in every way and with respect to all possible predicates".[467] This 'axiom' must now be abandoned. Likewise, we must revise our understanding of the whole–part relationship.[468] In particular, Cassirer notes, we must be careful to distinguish various possible meanings of 'inclusion' in this context. Thus we should distinguish 'containment' from 'being a part of'—a point is 'contained' in a line but should not be considered as forming a part of it. Since a whole may contain more than parts, the whole may be larger than the sum of its parts and this is exemplified by quantum mechanics, where a two-electron system determines the state of the electrons but not vice versa: "A knowledge

[465] The issue of moving from the kind to the particular within a structuralist framework is discussed in French 2001.

[466] For further discussion, see French 2002 and 2003.

[467] Cassirer 1956, pp. 188–9.

[468] Ibid., pp. 186–7.

of the states of the two parts does not determine the state of the joint system, and a derivation of the latter from the former is out of the question"[469]

The problem, then, is how such an aggregate may be "differentiated and 'individualized' ", and Cassirer notes, presciently perhaps, that

> The ordinary method of counting, which presupposes that it is known from the beginning what is to constitute one thing and what two or more things, is here insufficient. Individual things are not separated from each other in as simple a manner as in the sensuous-spatial view; complicated theoretical considerations are thus always required in order to determine precisely what is to be treated as an individual, what is to be counted as a 'one'.[470]

The formal analysis of aggregates was taken further by Weyl in his classic 1949 work, *The Philosophy of Mathematics and Natural Science*. Weyl begins by emphasizing the structuralist understanding of objectivity:

> Perhaps the philosophically most relevant feature of modern science is the emergence of abstract symbolic structures as the hard core of objectivity behind—as Eddington puts it—the colorful tale of the subjective storyteller mind.[471]

Among the 'simplest structures imaginable' are those which represent the combinatorics of aggregates and Weyl finds it gratifying that this 'primitive piece of symbolic mathematics' is so intimately related to philosophical issues having to do with individuality and probability, on the one hand, and fundamental physical phenomena in quantum mechanics and genetics, on the other.

Thus he begins by considering an aggregate of elements of the same *kind* and notes that this requires one to make the distinction we presented in Chapter 1, namely between *indistinguishability*, in the sense of possessing the same set of intrinsic properties, and *identity* per se. This raises the question of individuation, a question which, according to Weyl, Leibniz answered *a priori* with his Principle of Identity of Indiscernibles. An empirical answer, which is yet 'precise and compelling', is given by quantum statistics.[472] The question of trans-temporal identity is closely related to such issues but for the case under consideration, the decision as to what is to be counted as equal and what as different influences the counting of arrangements on which the calculation of probabilities rests and hence "the problem of individuation touches the roots of the calculus of probability".[473]

[469] Cassirer 1956, p. 187. Here Cassirer cites Weyl 1931.

[470] Ibid.

[471] Weyl, op. cit., p. 237.

[472] See also Weyl 1950, p. 241.

[473] Ibid., p. 238.

An aggregate can then be defined as a set $S$ (whose cardinal is $n$, for example), on which is defined an equivalence relation ~ such that elements which are in the same equivalence class are said to be of the same kind, or to be in the same state. Each element of $S$ is labelled by an arbitrarily chosen distinct mark, $p$, and if each such element is capable of being in $k$ classes, or states, then a definite 'individual state' of $S$ is given if it is known, for each of the marks $p$, to which of the $k$ states the element marked $p$ belongs.[474] This marking is subject to the 'principle of relativity' which states that the only statements and relations which have objective significance are those which are not dependent on any change in the choice of marks.

If it is known, for each of the $n$ marks $p$, to which of the $k$ classes the element marked $p$ belongs, a definite individual state of $S$ can be defined. However, if "… no artificial differences between elements are introduced by their labels $p$ and merely the intrinsic differences of state are made use of",[475] then $S$ is characterized by its 'visible' or 'effective' state obtained by assigning to each class the number $n_i$ of elements of the aggregate that belong to that class. Any two individual states are connected with the same effective state if and only if they are related by a permutation of the marks. In this way the above 'principle of relativity' comes to be manifested as a 'postulate of invariance' with respect to the group of permutations. Since it is the 'visible' state that is considered in quantum mechanics, it is obviously important to know how many different such states there are. This number equals that of the 'ordered' decompositions $n = n_1 + n_2 + \cdots + n_k$ of $n$ into $k$ summands $n_i$. And this has the value

$$\frac{(n+k-1)!}{n!(k-1)!} \tag{3.7.1}$$

that is, the number of ways of distributing $k$ ($P$) energy elements over $n$ ($N$) resonators, in Planck's expression, or of distributing $n$ (bosonic) particles over $k$ cells in Einstein's (from 1925).

Weyl then considers two processes the aggregate can effectively undergo. The first is a partition into two complementary sub-aggregates, where the number of distinct such partitions is $2^n$. Since the elements are now discernible according to their 'kind', a sub-aggregate, $S_1$, say, can be characterized by assigning to each class $C_i$ the number of elements, $n_i^1$, with which that class is represented in $S_1$. $n_i^1$ has the $n_i+1$ values 0, 1, … $n_i$ and hence the number

[474] See also Weyl 1950, p. 239. It should be remarked that the equivalence classes are pair-wise disjointed, so no element of $S$ can belong to more than one class.

[475] Ibid.

of different possible effective partitions of $S$ into such complementary sub-aggregates is $(n_1+1) \ldots (n_k+1)$.

This reaches a maximum of $2^n$ if all the $n_i$ have the value 0 or 1. This in turn would correspond to a situation where no two elements of $S$ are ever found in the same class, so one can effectively dispense with the labels. Weyl calls this a 'monomial' aggregate.[476] The second process is the inverse of partition, namely the union of two disjoint aggregates into a larger one.

So far Weyl's analysis has been conducted in entirely formal terms, but the application to quantum statistics is obvious.[477] A gas of either photons or electrons can be described in terms of an aggregate as given above, since the particles have no identity[478] and hence "[n]o specification beyond what was previously termed the effective state of an aggregate is therefore possible".[479] Thus the aggregates are characterized by giving, for each possible state, the number of elements (photons or electrons) in that state. The general case, as noted above, corresponds to Bose-Einstein statistics and the Fermi-Dirac form is obtained through the imposition of Pauli's Exclusion Principle, which, as Weyl notes, corresponds to the formation of a monomial aggregate (the two forms are related to the symmetry of the wave function for the aggregate in the following appendix).[480] He concludes by suggesting that "[I]n a profound and precise sense physics corroborates the Mutakallimûn (cf. footnote [221]); neither to the photon nor to the (positive and negative) electron can one ascribe individuality".[481]

We suggest that what Weyl was trying to do in this formal analysis of the kinds of 'aggregates' suitable for quantum physics, was to arrive at a form of set

[476] See also Weyl 1950, p. 240. See also Krause 1991.

[477] Ibid., pp. 245–7. What is also interesting, although we shall not go into it here, is Weyl's application of this formal analysis to genetics (ibid., pp. 240–5). Here the elements are the genes, of course, the different discrete states are the alleles, and union and partition then correspond to syngamy and meiosis respectively. Of more general interest is his discussion, in this context, of temporal change, where the principle of relativity forces a 'non-individualizing description' according to which we are told only how many elements are found in a state at a given time but not whether a particular element now in the state was earlier in some other given state. Weyl likens this picture—so obviously consonant with quantum field theory—to the metaphysics of the Islamic philosophers, the Mutakallimûn, who understood the world to be created anew at every moment, with no 'bond of identity' joining the elements across time.

[478] Ibid., p. 246.

[479] Ibid.

[480] Ibid., pp. 262–3. Here he identifies Pauli's principle with Leibniz's Principle of Identity of Indiscernibles and indeed refers to the 'Leibniz-Pauli' exclusion principle (ibid., p. 247). Pauli himself vigorously rejected such an identification, insisting that Leibniz's principle says nothing about whether two similar particles can occupy the same region of space (as in the case of photons) or not (as in the case of electrons); see von Meyenn 1987, p. 342. We shall analyse the status of the Principle of Identity of Indiscernibles in the quantum context later.

[481] Ibid., p. 247.

theory appropriate for elements without identity. His 'principle of relativity' effectively allows him to 'forget' the 'nature' of the elements of $S$ and by paying attention exclusively to the cardinality $n^i$ ($i = 1,\ldots, k$) of the equivalence class, he obtains the 'decomposition' $n_1 + \cdot + n_k = n$ (which notably bears a certain resemblance to the occupation numbers) which, as Weyl emphasizes, is what is considered in the quantum context. What this procedure does is to effectively 'mask' the individuality of the elements of the set $S$. Weyl's 'aggregates' therefore constitute kinds of sets in which the elements are initially labelled, but then by focusing on what remains invariant under the action of the permutation group, the formal effect of these labels is 'wiped out'. We see this as an early attempt at a form of 'quasi-set' theory and in subsequent chapters we shall suggest a way in which Weyl's programme can be completed and the Received View underpinned with a secure formal basis. Before we do so we need to complete our version of the history of quantum statistics, since later developments in this history have been turned to in order to support an alternative to the Received View. These developments concern more general forms of quantum statistics, known as 'parastatistics', and not only did they play quite a significant role in elementary particle physics in the 1960s, but further generalized forms are still under active consideration today.

## 3.8 BACK TO THE HISTORY: PARASTATISTICS

As we have indicated, as early as 1926 Dirac realized that the symmetric and anti-symmetric functions were merely the two simplest out of the set of possible eigenfunctions for the assembly. Thus he later wrote,

> It appears that all particles occurring in nature are either fermions or bosons, and those only anti-symmetrical or symmetrical states for an assembly of similar particles are met with in practice. Other more complicated kinds of symmetry are possible mathematically but do not apply to any known particles.[482]

That other mathematical possibilities exist was also, not surprisingly, noted by Weyl, who insisted that the situation in which these possibilities are not physically realized was 'repugnant' to Nature and wrote that,

> She has accordingly avoided this distressing situation by annihilating all these possible worlds except one—or better, she has never allowed them to come into existence! The

[482] Dirac 1978, p. 211.

> one which she has spared is that one which is represented by anti-symmetric tensors, and this is the content of Pauli's exclusion principle.[483]

The investigation into these alternative symmetry types was pursued along various lines. Perhaps the earliest can be traced back to Gentile, who, in 1941, used Bose's combinatorial methods to obtain an expression for the average number of particles in a group of states which was dependent upon a parameter $d$ giving the maximum number of particles which could occupy any given state.[484] Fermi-Dirac (F-D) and Bose-Einstein (B-E) statistics were special cases of these 'intermediate' statistics, obtained when $d = 1$ or $\infty$, respectively.[485]

Gentile's work represents a straightforward kind of generalization of standard quantum statistics. Two alternative lines of development focused on the implications of Dirac's claim above in the context of quantum mechanics (QM) on the one hand and quantum field theory (QFT) on the other. Before we pursue the historical details, we need to give a preliminary answer to the question, how can there be 'more complicated kinds of symmetry' than the usual symmetrical and anti-symmetrical forms? Within the formalism of QM, the physical state of a system is represented by a vector, or basis function, spanning a subspace of Hilbert space. If we have a system of $N$-particles, then the relevant subspace will have $N!$ dimensions and will be spanned by the vectors obtained by permuting the particles in the system (we shall return to consider the nature of these permutations shortly). This subspace can then be decomposed into irreducible subspaces each carrying an irreducible representation of the group of permutations. The symmetric and anti-symmetric vectors, corresponding to Bose-Einstein and Fermi-Dirac statistics respectively, span one-dimensional subspaces, related to one-dimensional representations of the permutation group. However, representations of higher dimension are also possible and correspond to higher dimensional subspaces of the relevant Hilbert space spanned by multi-dimensional 'rays'. It is these which represent more general forms of statistics, known as parastatistics.

[483] Weyl 1968, p. 288.

[484] Gentile 1940.

[485] This work was criticized—first by Sommerfeld in 1942 and subsequently by ter Haar in 1952 (Sommerfeld 1942; ter Haar 1952)—on the grounds that when $d = N$, the number of particles in the system, there would be no difference between Gentile's statistics and the Bose-Einstein form. By an inspection of the relevant integrals in the complex plane, Wergeland and Schubert concluded that it actually did not matter whether one took $d$ to be equal to $N$ or $\infty$. Furthermore, the condition of having at most $n$ particles in a given state does not remain invariant under change of basis on Gentile's theory (see Greenberg 1992). ter Haar's other criticism, that Gentile's work was in conflict with the fundamental postulates of quantum statistics, founders on his assumption that the relevant wave functions can be symmetric or anti-symmetric only.

Although this possibility was clearly perceived by Dirac and Weyl, the theoretical exploration of these parastatistics can be traced back to a little-known work published by Okayama in Japan in 1952.[486] Okayama anticipated later discussions in his demonstration that the standard statistics only follow if the relevant subspace is assumed to be one-dimensional and that if this assumption is dropped, and the possibility allowed of a single state corresponding to some larger collection of vectors spanning a multi-dimensional subspace, then other forms of statistics can be admitted. And in particular, he emphasized that these alternative forms are related to various representations of the permutation group and could hence be investigated using the apparatus of group theory.[487] Okayama also attempted to develop a second quantized approach to paraparticle theory, but his generalized commutation relations allowed only the trivial solution that the Hilbert space consists only of null vectors.[488]

A much more successful and more well-known attempt to construct a form of parafield theory was made by Green in 1953, following Wigner's demonstration that the standard commutation relations were not uniquely determined by the equations of motion.[489] Green's aim in this work was to see if certain fundamental problems with quantum field theory could be resolved by relaxing the theory's formal structure.[490] By regarding any quantization scheme to be deemed satisfactory if it simply ensured the equations of motion, Green was able to introduce whole families of alternative commutation schemes obeyed by paraparticles. Each such scheme can be labelled by an integer $p$ and following Green's work, if one quantizes with the $p$'th scheme, one talks of a parafermion or paraboson field of order $p$. When $p = 1$, one recovers the standard bosonic and fermionic schemes, of course. From the perspective of the themes we are interested in here, it is interesting that Green argued that although his generalized theory implied that a new state would result from the interchange of two particles, the particles would always divide up into groups such that permutations within a group would not lead to a new state and furthermore, that interactions can be devised which prevent the formation of new states by permutations *between* the groups. This allowed him to conclude that the 'Principle of Indistinguishability of Identical Particles' would be retained in parafield theory.[491]

[486] Okayama 1952.

[487] Ibid., pp. 517–21.

[488] See the comments in Kamefuchi and Takahashi 1962.

[489] Green 1953.

[490] It is a matter of some historical curiosity that Green's only other work—so far as we know—in parafield theory did not appear until 1972.

[491] The physical properties of Green's paraparticles were further explored by McCarthy (McCarthy 1955). Over the next few years it was shown that Green's parafield theory was consistent, both internally and with the fundamental principles of both relativistic and non-relativistic physics (see Kibble and Polkinghorne 1957; Kamefuchi and Takahashi op. cit.; Dell'Antonio, Greenberg and Sudarshan 1964) and that its formalism could be extended to allow for interactions between fields (see Kamefuchi and Strathdee 1963).

Despite all this theoretical work, there did not appear to be any evidence for the existence of paraparticles in nature. This issue was addressed by Greenberg and Messiah in their famous construction of a consistent first-quantized para-particle theory.[492] By this time—the early 1960s—Heisenberg's and Dirac's crucial insight that the wave function for an assembly of particles should be either symmetric or anti-symmetric had become enshrined in the so-called 'Symmetrization Postulate' (SP). Greenberg and Messiah pointed out that this is "… an extremely strong condition, very much stronger than what is implied by the indistinguishability of identical particles".[493] In its place they proposed a weaker 'Indistinguishability Postulate' (IP). This states that if the particles of the assembly are permuted, then there is no way of observably distinguishing the wave function which results from the original unpermuted wave function.[494] We shall consider IP in more detail, both formal and philosophical, in the next chapter but for the moment we simply wish to note Greenberg and Messiah's observation that SP is sufficient but not necessary for IP. The former can be regarded as a restriction on the states for all observables—namely that they must be suitably symmetrized—whereas the latter can be understood as a restriction on the observables for all states—namely that they must commute with the operator representing a particle permutation. Greenberg and Messiah went on to insist that the usual ways of introducing SP into the quantum mechanical formalism were suspiciously *ad hoc* in nature and that many experimental results which had been taken to be tests of SP were actually just tests of the weaker IP.[495]

Their overall approach to this issue was to take the usual assumption in quantum mechanics that the wave functions are represented by one-dimensional state vectors in Hilbert space and generalize it, to allow states to be represented by vectors spanning a multi-dimensional subspace of the Hilbert space. They called the set of vectors spanning such a multi-dimensional subspace a 'generalized ray' and showed that the indeterminacy associated with such a move (since there is now more than one vector associated with a given state) actually causes no difficulty in the interpretation of the theory because measurable results do not depend on precisely which state

[492] op. cit. The name 'parastatistics' was coined by Dell'Antonio, Greenberg and Sudarshan in 1964.

[493] Ibid., p. 248.

[494] Ibid., p. 250.

[495] Ibid., pp. 259–67.

vector spanning the multi-dimensional subspace is chosen to represent the state.[496] Furthermore—and this will be important for our discussion in Chapter 4—Greenberg and Messiah demonstrated that the Indistinguishability Postulate (together with a formal result known as Schur's Lemma) imposes what is known as a 'superselection rule' on Hilbert space which effectively decomposes it into a number of subspaces such that once a particle is in a particular subspace—corresponding to bosonic, fermionic, or some form of paraparticle statistics—the particle must remain in that subspace.

Greenberg and Messiah also made it clear that those arguments which purport to demonstrate that state vectors must be either symmetrical or anti-symmetrical involve an implicit assumption to the effect that states which cannot be distinguished by any observation must be represented by the same vector to within a phase factor.[497] The introduction of 'generalized rays' shows how such an assumption can be abandoned while retaining the Indistinguishability Postulate.[498] Hartle and Taylor subsequently showed how Greenberg and Messiah's generalized rays could be eliminated and the usual connection between states and rays restored by moving to a subspace of lower dimension and using the one-dimensional ray belonging to such a subspace to label the mixed symmetry states.[499]

In the second half of their paper, Greenberg and Messiah gave a very detailed discussion of the direct experimental tests of the Symmetrization Postulate and concluded that in many cases what such experiments actually tested was the Indistinguishability Postulate (IP) instead.[500] After surveying the experimental evidence for SP for various kinds of particles they concluded that although their statistical characters were accepted and well established, there were still some for which it was not entirely clear what form of statistics they should obey.

This opened up the possibility that the behaviour of some particles might be described by parastatistics. Hartle and Taylor may have expressed the views of many physicists when they wrote, "… although there is no theoretical reason to exclude para-particles, their properties are sufficiently disagreeable for one to hope sincerely that there will continue to be no evidence in their favor".[501] Nevertheless, in a famous and historically important paper

[496] Ibid., p. 251.

[497] Ibid., p. 253.

[498] This allowed the theory to meet the criticism of Galindo and Yndurain, for example, that paraparticles cannot be regarded as 'indistinguishable', in the physicists' sense (see Galindo and Yndurain 1963). The latter effectively assumed that state vectors must transform under a one-dimensional representation of the permutation group only.

[499] Hartle and Taylor 1969.

[500] op. cit., pp. 259–67.

[501] op. cit., p. 251.

Greenberg suggested that the newly introduced quarks might be regarded as paraparticles of a certain type.[502] However, it was subsequently shown that this was equivalent to the introduction of a three-fold degree of freedom for quarks that came to be called colour and which led to the development of quantum chromodynamics.[503] Following the demonstration that the colour model could be made gauge invariant, whereas the paraquark model could not, interest in paraparticle theory declined, at least among mainstream physicists.[504]

There still remains the question of the relationship between Greenberg and Messiah's theory of paraparticles and Green's parafield theory. The answer is somewhat delicate since, as Greenberg had emphasized, a permutation of the particle labels cannot be represented within Quantum Field Theory where, on the standard formulation, such labels are not introduced to begin with. This meant that IP is ill-defined within Quantum Field Theory. However, this problem can be overcome through the use of 'state' or 'place' permutations which act on the indices labelling the different states and which *are* well-defined in parafield theory. This distinction goes back to Dirac,[505] who noted that, unlike the particle permutation operators, the place permutation operators depend upon the basis with respect to which they are defined. Although the difference between these two kinds of operation does not emerge for two particles distributed over two states (one might think of permuting two balls between boxes or permuting the boxes around the balls, for example) and is generally not significant for ordinary statistics, it is apparent once we consider three particles and the possibility of parastatistics; in particular, it turns out to be crucial for establishing the relationship between paraparticle theory and parafield theory.

[502] Greenberg 1964. According to Ne'eman, it was the discovery of the $\Omega^-$-particle,-particle whose (quark) wave function was symmetric in spin, isospin and orbital angular momentum, that led to parastatistics being taken seriously. He recalls that he and Gell-Mann read an early paper by Kamefuchi but dismissed it when they discovered a mistake. A few weeks later Greenberg published his paper, based on work which apparently did not contain the same mistake (Ne'eman 1987, p. 549). Gell-Mann remembers things slightly differently: "There was not only a mistake, as I remember. We also were unsatisfied in that we did not see any way to eliminate parabaryons while leaving regular baryons" (Gell-Mann 1987, p. 549). This issue remained unresolved until 1971, when Gell-Mann realized that the parastatistics model with suppressed paraparticles was equivalent to the colour model with suppressed quarks (so the parabaryons would be suppressed). "That led directly to our work on quantum chromodynamics with color suppression" (ibid.).

[503] Han and Nambu 1965; see also Greenberg and Nelson 1967.

[504] French 1995. Gauge invariance is a particularly important symmetry principle which was effectively imported from quantum electrodynamics.

[505] op. cit.

Thus it was demonstrated that IP is equivalent to the requirement that all observables that distinguish among states differing only in the order of the variables are functions of the place permutation operators.[506] On this basis the equivalence between parafield and paraparticles of 'finite order' could be established.[507]

With the Indistinguishability Postulate at the heart of paraparticle theory, the shift back to a field theoretic perspective allowed for the possibility of further theoretical generalizations. Kamefuchi and Ohnuki, in particular, embarked upon a general group theoretical investigation of 'indistinguishability', or loss of identity in this context.[508] They subsequently went on to consider the necessary and sufficient conditions for particles to be considered indistinguishable (in the quantum sense). Arguing that the premises used to establish IP were too restrictive, they adopted the following field-theoretic condition: those particles are regarded as indistinguishable which result from the second quantization of a given field, since they share the same intrinsic properties.[509] The quantum mechanics of such particles then follows from the corresponding field theory and Kamefuchi and Ohnuki obtained a general theory of indistinguishable particles by simply translating the field theoretic results concerning many particle systems into the language of quantum mechanics. This formed the basis of a further generalization covering forms of statistics different from both ordinary and parastatistics.[510] The exploration of such forms of non-standard statistics has continued to the present day and has proceeded in both experimental and theoretical directions.[511]

At this point we need to stop. Our central aim in this chapter was to sketch the emergence of the Received View of particle non-individuality in its historical context. As we hope to have illustrated, a concern with particle indistinguishability runs through the history of quantum statistics like a thread,

[506] Landshoff and Stapp 1967.

[507] Stolt and Taylor 1970. A particle is said to have finite column/row order $p$ if it possesses states associated with some Young diagram of $p$ columns/rows, respectively, but no states associated with any such diagram of more than $p$ columns/rows. If a particle has neither finite row nor column order then it must have states with diagrams of arbitrarily many rows and columns; such a particle is said to be of infinite order. Finite order paraparticles can be further divided into parabosons and parafermions of order $p$, a division which closely resembles that of parafield theory (Stolt and Taylor op. cit., p. 2228). A similar classification can be established for infinite order paraparticles, except those with states corresponding to all Young diagrams.

[508] Kamefuchi and Ohnuki 1968 and 1969.

[509] Ohnuki and Kamefuchi 1970.

[510] See Ohnuki and Kamefuchi 1982; cf. also the work of Carpenter, following Landshoff and Stapp's work; Carpenter 1970.

[511] See, for example, Greenberg and Mohapatra 1987 and 1989; cf. Greenberg 1990; see also Hilborn and Yuca 2002, pp. 35–40.

connecting Ehrenfest's puzzlement over Planck's non-classical counting with Kamefuchi and Ohnuki's analysis of paraparticle theory. We have included the later material on parastatistics because this offers a broader formal framework in which to represent this notion of indistinguishability. As we shall see in the next chapter, the Indistinguishability Postulate offers a useful basis on which to explore the philosophical implications of quantum statistics in more detail. In particular, it opens up the possibility of an alternative metaphysics to the Received View, one in which quantum particles can still be regarded as individuals.

# 4
# Individuality and Non-Individuality in Quantum Mechanics

> You cannot mark an electron, you cannot paint it red. Indeed, you must not even *think* of it as marked.
> E. Schrödinger 1953

Let us recall our discussion from Chapter 1 of the Principle of Individuality. The following broad positions were set out: on the one hand there are those views which strive for a kind of metaphysical economy by advocating a Principle which serves double-duty in grounding both individuality and distinguishability; on the other, there are those positions which are rooted in the insistence that the two must be kept distinct, at least conceptually. Of the former, perhaps the most straightforward, at least initially, is the view that attempts to ground both distinguishability and individuality in the properties of objects. This requires a kind of 'guarantor' of individuality such as the Principle of the Identity of Indiscernibles (PII) which, we recall again, states that two things simply cannot possess the same set of (relevant) properties. The debate then shifts to the status of this Principle—whether it is necessarily true or only contingently so, and so on. As we saw, certain forms of PII can be maintained in classical mechanics. In particular, if we assume some form of impenetrability—as Newton did, for example—then no two things can have exactly the same spatio-temporal properties and if the latter are included in our set, then PII(1) appears to hold. Spatio-temporal location then performs a nice unifying role in our metaphysics: it allows us to distinguish even those things which share all their other properties—and in terms of which these things are indistinguishable—and it acts as a Principle of Individuality.

The alternative tradition insists that distinguishability and individuality must be kept distinct. Accordingly, what makes a thing an individual cannot be any multiply instantiable property—these relate to distinguishability only—but rather something else, over and above or transcending such properties. According to some, this something else is what the Scholastics called a thing's 'haecceity', reincarnated in modern times as its 'primitive thisness' which has been further explicated in terms of self-identity ('$a = a$'). According to others, it is substance, famously characterized as 'something we know not what'. Those of a broadly empiricist persuasion have not taken too kindly to such notions, as we indicated, but they have proven notably resilient and hard to do away with.

Our intention in this chapter is to further examine the impact of quantum mechanics on these positions. We have already indicated how this impact was understood historically, by those engaged in the elaboration of the new statistics, namely that quantum particles, unlike their classical counterparts, cannot be regarded as individuals. As we shall see, this conclusion does not necessarily follow from the physics.

# 4.1 INDISTINGUISHABILITY AND INDIVIDUALITY

We shall begin by recalling the argument that leads to the conclusion that quantum particles are non-individuals, in some yet to be worked out sense. The connection between the above philosophical views and quantum mechanics was expressed very clearly by that great philosophical commentator on modern physics, Reichenbach (responding, we suggest, to physicists' reflections on this issue as sketched in the previous chapter). He argued that the impact of modern physics on the philosophy of individuality could be evaluated by examining the behaviour of the fundamental particles of physics in aggregate.[512] The examination proceeds in terms of the permutation argument presented in Chapter 2: we consider the distribution of indistinguishable particles over states—two particles over two one-particle states, say—and assume that each resulting arrangement is accorded equal probability.[513] This

[512] Reichenbach 1956.

[513] It is worth noting at this stage that in the quite extensive literature on this topic, terms such as 'indistinguishable', 'identical', 'indiscernible' etc., are used in different ways by different people. Here we shall begin our discussion with the understanding of 'indistinguishable' which we have previously elaborated, namely two particles are indistinguishable if they share all intrinsic, state-independent properties in common. However, as we shall soon see, a stronger sense of indistinguishability can be elaborated in the quantum context, one which can be interpreted either as undermining particle individuality entirely, or, at the very least, as rendering it more problematic than in the classical case.

generates four possibilities:(1) particles 1 and 2 in state $|a^1\rangle$

(2) particles 1 and 2 in state $|a^2\rangle$

(3) particle 1 in state $|a^1\rangle$ , particle 2 in state $|a^2\rangle$

(4) particle 1 in state $|a^2\rangle$ , particle 2 in state $|a^1\rangle$ .[514]

We have labelled the states with superscripts 1, 2,…. This reflects the fact that the states can be distinguished and assigned labels in this way because different states are characterized by different state functions. A second set of labels can now be introduced which help to specify which particles are in which states.[515] These are the particle labels, represented by subscripts and hence (1)–(4) above can be rewritten as:

(1′) $|a_1^1\rangle \otimes |a_2^1\rangle$

(2′) $|a_1^2\rangle \otimes |a_2^2\rangle$

(3′) $|a_1^1\rangle \otimes |a_2^2\rangle$

(4′) $|a_1^2\rangle \otimes |a_2^1\rangle$

The introduction of particle labels here might be regarded as philosophically more contentious than in the case of the states, but we shall leave consideration of this until Chapter 5.

In classical statistical mechanics, as we saw in Chapter 2, (3) and (4) (or (3′) and (4′)) are counted as distinct and given equal weight in the assignment of probabilities; that is, the situation where we have one particle in each state is given a weight of two, corresponding to the two arrangements or complexions that may be formed by a permutation of the particles.[516] That a permutation of the (indistinguishable) particles is included in the count of possible arrangements is taken to imply that the particles are individuals, spelled out in terms of some underlying 'haecceity' or 'primitive thisness' or, more typically, the spatio-temporal location of the particles.

Now we can see how the examination is going to proceed in the quantum case. We shall initially restrict our attention to the two standard forms of 'statistics'—Bose-Einstein and Fermi-Dirac. In both cases the arrangement of

[514] Here we use the standard 'bra' and 'ket' notation for describing quantum states; see Dirac 1978.

[515] Of course we are assuming here that the particles are associated with the relevant places or 'slots' in the tensor product. This seems a natural assumption to begin with, at least.

[516] Alternatively one could adopt Huggett's strategy, as outlined in Chapter 2, but as we indicated, the statistics still requires the attribution of some form of individuality.

one particle in each state is given a weight of one; that is, in order to get the correct results in quantum statistics (3) and (4) (or, again, (3') and (4')) must be counted as one and the same.[517] This is standardly taken to reflect the fact that arrangements obtained by a permutation of the particles do not feature in the relevant counting in quantum statistics.

Let us express this point a little more formally. For simplicity we consider only one kind of spinless particle and we keep the number of particles fixed. We further assume that the one-particle states are in one-to-one correspondence with the rays of the Hilbert space $\mathscr{H}$ made up of all wave functions of one coordinate. For many particle systems the Hilbert space is the joint space constructed by forming the tensor product of the component particles' Hilbert spaces. The observables $\hat{O}$ of a quantum system are then represented by Hermitian operators acting upon that system's Hilbert space.

Now consider a system consisting of two indistinguishable particles as above. The Hilbert space for this system is: $\mathscr{H}_{total} = \mathscr{H}_1 \otimes \mathscr{H}_2$, where the subscripts '1' and '2' label the component particles, and $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$. Given the state of the system represented by (3'), for example, another state can be obtained by permuting the particles, where for our two-particle system this state is represented by (4'). This process of permuting the particles can be represented by a unitary operator which acts on the particle labels and these 'permutation operators' form a group known as—surprise, surprise—the Permutation Group.[518]

That particle permutations are not counted is then understood in terms of there being no measurement that we could perform which would result in a discernible difference between permuted (final) and unpermuted (initial) states. This is expressed by the so-called 'Indistinguishability Postulate' (IP):

> if a particle permutation is applied to any ket for an assembly of particles, then there is no way of distinguishing the resultant permuted ket from the original unpermuted one by any means of observation at any time.[519]

It can be represented more formally by insisting that every physical observable $\hat{O}$ must commute with every permutation operator $\hat{P}$: $\left[\hat{O}, \hat{P}\right] = 0$, $\forall \hat{O} \forall \hat{P}$. In other words, we have that for any arbitrary state $\psi$, Hermitian

[517] We are making the usual assumption here that (1')–(4') correspond to unique physical states. We could, of course, obtain the right statistics in the Bose-Einstein case, say, by counting (1') and (2') as each corresponding to two possibilities, but this would be bizarre.

[518] See, for example, Weyl 1931. In general for $N$ particles, the application of these operators will yield $N!$ states, corresponding to the permutation of $N$ particle labels among themselves.

[519] Greenberg and Messiah 1964, p. 250.

operator $\hat{O}$, and permutation operator $\hat{P}$:

$$\langle\hat{\psi}|\hat{O}|\psi\rangle = \langle\hat{P}\psi|\hat{O}|\hat{P}\psi\rangle = \langle\psi|\hat{P}^{-1}\hat{O}\hat{P}|\psi\rangle \tag{4.1.1}$$

The most significant observable of a quantum system is, of course, the Hamiltonian, $\hat{H}$, and the indistinguishability of the particles thus requires that this be a symmetric function of the dynamical variables;[520] that is, $\hat{H}$ is invariant under the action of the permutation group of permutations of the composite particles' labels: $\left[\hat{H},\hat{P}\right]=0$, $\forall\hat{P}$.

As we indicated in Chapter 3, (IP) allows for the possibility of forms of quantum statistics which are different from the 'standard' Bose-Einstein and Fermi-Dirac kind. If one wants to restrict the formalism to the latter kinds only, then a further condition, known as the 'Symmetrization Postulate' (SP) must be applied,[521] as we noted. Put simply, this dictates that states of indistinguishable particle systems must be either symmetrical or anti-symmetrical under the action of the permutation operators (corresponding to the Bose-Einstein and Fermi-Dirac cases respectively). The difference between SP and IP can be expressed as follows: SP expresses a restriction on the states for all observables, $\hat{O}$ in the first place; whereas IP expresses a restriction on the observables, $\hat{O}$, for all states.[522]

Now, IP seems to run counter to the whole point of regarding the particles as individuals and labelling them—from the point of view of the statistics, the particle labels are otiose. The implication, then, is that the particles can no longer be considered to be individuals, that they are, in some sense, 'non-individuals'. This conclusion expresses what we have called the 'Received View': classical particles are individuals but quantum particles are not. We have indicated how this view emerged, historically, in Chapter 3 and it achieved further prominence in the 1960s through the work of Hesse and Post.[523] Hesse attempted to express the relevant metaphysics metaphorically, in terms of money in a bank account, for example: I can say that I have £300 in my current account but I can't go in, as it were, and point to a particular £ and say, 'that's mine'. More recently Teller[524] has resorted to the same metaphor in order to explicate how something might no longer be considered an individual.

[520] See, for example, Dirac op. cit. p. 207.

[521] Messiah 1962, p. 595.

[522] We have said that SP restricts states 'in the first place' because it is not correct that it restricts only states. The action of a generic observable of the full tensor product space on a vector in the symmetric space is not another vector in that space and so not all observables of the full space are well-defined in the symmetric space. Again, we are grateful to one of the referees for pointing this out.

[523] Hesse 1963; Post 1963.

[524] Teller 1995.

Post, on the other hand, drew on the distinction between form and substance, arguing that what quantum statistics indicates is the ontological primacy of the former over the latter. As we have indicated, both the metaphor and the distinction can be traced back to Schrödinger's reflections on the implications of the new quantum mechanics, And as we shall see in the rest of the book, one can in fact go beyond mere metaphor and underpin the Received View with an appropriate logico-mathematical framework. Before we do that, however, we need to consider two broad challenges it must face. Both challenges resist the argument for the existence of a fundamental metaphysical distinction between classical and quantum particles but approach the issue from different directions.

### 4.1.1 Challenge No. 1: Classical Particles as Non-Individuals

The first challenge accepts that quantum particles are non-individuals but rejects the claim that this marks a crucial difference from their classical counterparts. Thus, it has been suggested that classical particles of the same kind, like their quantum counterparts, not only possess the same set of intrinsic properties, but are also 'indistinguishable' in the stronger sense indicated above; that is, they are what we have called 'non-individuals'. Various arguments have been given for what at first sight appears such an unlikely claim. One such insists that since there is no provision for classical particles to carry any 'identification marks' they cannot be considered to be individuals either.[525] Now, as we have seen, if we consider non-spatio-temporal properties only, which might include such 'marks', and if the Identity of Indiscernibles is taken to ground our understanding of individuality, then it is indeed true that classical particles violate some forms of this Principle. On this basis they might then be regarded as non-individuals. However, we don't even have to fall back to some form of haecceity or some substantivalist metaphysics to save individuality in this case: we can simply invoke the relevant set of spatio-temporal properties, together with the Impenetrability Assumption, and come up with an appropriate form of the Identity of Indiscernibles which is not violated. As we shall see, however, this option is not available for quantum particles (at least not under the standard interpretation of quantum mechanics).

[525] Sudarshan and Mehra 1970.

An alternative argument explicitly adopts a broadly positivistic approach to particle individuality by insisting that the meaning of quantum 'indistinguishability', as expressed by IP, is determined experimentally.[526] Such non-individuality is then understood as not restricted to quantum particles, since it is taken to follow from the requirement that entropy be 'extensive'. We recall from Chapter 2 that this is the requirement that if the volume of some system (a gas, say) is divided into two halves, then the entropy of the whole system must equal the sum of the entropies of the respective halves. As we noted there, this requirement leads to the Gibbs Paradox.

The 'paradox' is resolved by simply dividing the classical expression for the number of possible complexions by N!. This is justified on the grounds that this procedure eliminates 'redundant' complexions obtained through a permutation of the particles. Now, how are we to interpret this procedure? One suggestion, following on from the above work which attempts to ground non-individuality on experimental considerations, is that it implies that we should regard classical statistical mechanics as subject to IP as well.[527] In that case, what we have up until now understood as 'classical' particles would also be viewed as 'non-individuals' and the contrast with quantum physics would have to be sought elsewhere. However, this suggestion radically misconstrues the historical situation and, by blurring the ontological character of classical particles, it also blurs the distinction between classical and quantum statistical mechanics in general. There is a more plausible alternative.

Historically, the failure of extensivity and the Gibbs Paradox in general were seen as revealing a fundamental flaw in the Maxwell-Boltzmann definition of entropy and one which is corrected by shifting to quantum statistical mechanics at the theoretical level and an understanding of the particles as *quantum* in nature at the ontological level. It is in this sense that quantum mechanics, in general, has been said to 'resolve' the problem.[528] In other words, the force of the argument can be turned around: what it shows is that the world is actually quantum in nature, as one would expect. What the exclusion of the permutations (by dividing by N!) is a manifestation of is *precisely* that the particles are not just indistinguishable in the classical sense. If this aspect is incorporated into the analysis from the word go, the so-called 'paradox' simply

[526] Hestenes 1970; Rosen 1964.

[527] See, e.g., Saunders forthcoming p. 22, fn. 13. There is a connection with Huggett's approach to classical statistical mechanics, insofar as this suggestion also sees the difference between classical and quantum statistics as residing in something other than counting permutations.

[528] See, e.g., Boyer 1970.

does not arise. As the author of an undergraduate textbook on statistical physics puts it:

> Historically, failure to appreciate the significance of the identity of the particles in a system led to certain inconsistencies, known as the Gibbs paradox. We shall not discuss this paradox since a careful reader of this book should not be troubled by it.[529]

Of course, this counter-suggestion is not decisive but one might wonder how, if classical statistics is to be subject to IP and classical particles regarded as non-individuals, classical statistics and particles respectively can still be called 'classical'? Once the division by N! is introduced (yielding the Sackur-Tetrode equation for the entropy, as we noted in Chapter 2), the effect can be thought of as rippling through the structure of classical statistical mechanics, effectively transforming it into the quantum form. In the process we appear to lose the very distinctions on which our discussion was originally based. But of course, if the above argument were to be accepted, this would provide further motivation for our programme of explicating the sense of 'non-individuality', in both the classical and quantum cases.

### 4.1.2 Challenge No. 2: Quantum Particles as Individuals

The second challenge accepts that classical particles are individuals, but resists the Received View by denying that it follows from quantum statistics—and, more particularly, the Indistinguishability Postulate—that quantum particles are non-individuals, in whatever sense. The conclusion expressed by the Received View depends on understanding IP simply as imposing restrictions on the set of possible observables, such that particle permutation operators cannot be included. It is this formal 'fact' which motivates commentators to adopt the metaphysics of non-individuality. However, it can also be shown that the action of IP is to divide up the relevant Hilbert space into a number of irreducible subspaces, corresponding to different symmetry types—Bose-Einstein, Fermi-Dirac, parabosonic, parafermionic and so on. These subspaces correspond to different irreducible representations of the permutation group. Thus, for example, in the case of three particles, the relevant six-dimensional subspace decomposes into irreducible subspaces invariant under the particle permutation operators, giving a one-dimensional symmetric subspace, another one-dimensional anti-symmetric subspace and

[529] Mandl 1971, p. 131.

two two-dimensional paraparticle or 'triangular' subspaces. The states for 'ordinary' particles correspond to the one-dimensional subspaces, whereas paraparticle states correspond to the multi-dimensional subspaces.[530]

Furthermore, given IP, it can be shown that all matrix elements of an observable connecting different representations are zero.[531] Hence, if there exist states corresponding to subspaces of different representations then transitions between such states must be forbidden.[532] In other words, bosons 'live' in the one-dimensional symmetric subspace, fermions occupy the anti-symmetric subspace, which is also one-dimensional and paraparticles remain in the multi-dimensional subspaces.

This gives us an important perspective on IP from which it can be seen as imposing a restriction on the states of the assembly of particles such that once a particle is in a given subspace, the other subspaces—corresponding to other symmetry types—are inaccessible to it. If we consider the time evolution of the system as effected by some Hamiltonian, then IP implies that if the system starts in a subspace corresponding to a particular symmetry type,[533] then it will always remain in that subspace: bosons will always be bosons, fermions will always be fermions etc. Once a system starts to evolve in a particular subspace, the dynamics is such that it can never get out of it. From this perspective, IP can be thought of as an extra postulate, representing an accessibility constraint on the set of states of the assembly.

If we go back to that simple example of two particles distributed over two states, this alternative understanding of IP yields a completely different explanation for the reduction in statistical weight: if, in this case, the restriction is imposed that the state of the system be either symmetric or anti-symmetric then only one of the two possible states formed by a permutation of the particles is ever available to the system and so the statistical weight corresponding to the distribution of one particle in each (one-particle) state is half the classical value. In the general case, where we have more than two particles, we get more general symmetry types, as we indicated, corresponding to parastatistics (and to accommodate these types we impose IP rather than simply SP). Each such type will correspond to a respective vector of the

[530] These are represented by the 'generalized rays' introduced by Greenberg and Messiah.

[531] French 1985, p. 161 and p. 170ff. Hilborn and Yuca call this the 'Fundamental Theorem' because of its importance in the context of possible violation of the spin-statistics theorem (Hilborn and Yuca 2002, p. 30).

[532] States corresponding to subspaces of the same representation are not separated in this way and for certain kinds of three-body interactions, transitions may occur between them (see French 1987).

[533] Or more formally, a particular irreducible representation of the permutation group.

Hilbert space, which will then be inaccessible to particles of a different symmetry type. From this perspective, states formed by a particle permutation are not counted not because they do not exist, but because they are simply not available to the particles of the relevant symmetry type.

With the reduction in statistical weight now explained by the inaccessibility of certain states, rather than by the non-classical metaphysical nature of the particles as non-individuals, one can continue to regard them as individuals for which certain states are now inaccessible—just because the particle labels are statistically otiose does not mean they are metaphysically so.[534] Quantum statistics is effectively recovered by regarding such states as possible but never actually realized.[535] Through this alternative explanation, this way of understanding IP allows us to retain the metaphysics of individuality for quantum particles. What is now doing all the work, as it were, is the understanding of IP as an extra requirement concerning state accessibility.

The two components of this view—the accessibility constraint and the individuality of the particles—can then be understood philosophically in ways that emphasize the commonality with the classical situation. Let us consider these components in a little more detail.

### 4.1.3 The Indistinguishability Postulate as an Initial Condition

The notion of state accessibility restrictions can, of course, also be found in classical statistics, albeit in a somewhat restrictive form. The most important and most obvious has to do with the available energy, but other uniform integrals of the equations of motion may exist for a particular assembly which have the effect of restricting its representative point to a certain region of phase space. With regard to thermodynamic consequences, only the energy integral is deemed significant and as imposing constraints on accessible regions of the phase space. In quantum statistics, we have these further constraints and the symmetry type of any suitably specified set of states is an absolute constant of motion equivalent to an exact uniform integral in classical terms.[536]

Hence this notion of accessibility can be formulated for both classical and quantum systems but the considerations of the previous section go some way toward explicating the greater role played by symmetry in the quantum case as

[534] We shall return to the status of the particle labels in the next chapter.

[535] For further details see French 1989a, French and Redhead 1988, Redhead and Teller 1992.

[536] Dirac op. cit., p. 213.

compared to the classical.[537] When it comes to ordinary statistics, the question as to what states are accessible is straightforward. In the paraparticle case it is slightly more complicated: for particles obeying Gentile's paragas statistics, the number of accessible states will simply be intermediate between the number accessible to fermions and bosons. This number will increase as the statistics becomes less fermionic and more bosonic. In the case of parastatistics, it turns out that it is a linear combination of states that now becomes accessible to the particles.[538]

## 4.2 THE INDIVIDUALITY OF QUANTUM PARTICLES

As we have suggested, this option also has the obvious advantage of allowing us to retain a classical ontology, at least where it concerns the individuals themselves. Perhaps that is its only advantage but it is a significant one. Indeed, it is sometimes claimed that such a view draws support from the very practice of experimental physics itself, with its individual tracks in a bubble chamber, distinct clicks from a counter and individual flashes on a scintillation screen.[539] There may be problems with regard to issues of reference etc., which we shall discuss in the next chapter, but whatever the outcome of that particular philosophical discussion, quantum particles can be considered as individuals, just like classical particles, chairs, tables and people. This meshes very nicely with that very general approach to quantum mechanics which seeks to interpret the theory in terms that, ontologically speaking, differ as little as possible from classical mechanics (or, better, classical statistical mechanics). Thus it could serve as the underlying ontology of some sort of hidden variables approach in general or, more particularly, of the Bohm-Hiley interpretation, where you have individual particles chugging along well-defined spatio-temporal trajectories (we shall touch on such accounts below). Of course, the peculiarities of quantum mechanics will then emerge elsewhere in the metaphysics. For the moment let us focus on the individuality of the particles and, as we have seen, there are a range of options, some more attractive than the others. We shall

[537] For further discussion of IP as a symmetry principle, see French and Rickles 2003.

[538] As we indicated above, in this case there is a possibility of transitions occurring between states carrying the same representation (i.e. between the 'triangular' subspaces).

[539] Falkenburg suggests that this claim can be construed as forming the basis of a kind of 'transcendental argument' for particle individuality (Falkenburg 1996). It can certainly be understood as the basis for Toraldo di Francia's notion of 'pseudo-individuality'. How such a view can then be made to cohere with the formalism of the theory lies at the heart of our discussion to follow.

begin with what is metaphysically perhaps the most straightforward view: both individuality and distinguishability are grounded in some (sub-)set of properties of the particles. In order to guarantee individuation, as we have expressed it, such a view requires some form of the Principle of Identity of Indiscernibles (PII) but the viability of this principle is extremely problematic in the quantum context, as we shall now see.

## 4.2.1 Quantum Mechanics and the Principle of Identity of Indiscernibles

A great deal has been written about the status of PII in quantum mechanics, beginning with Weyl, back in 1928, as we saw in the last chapter. His work represents the origin of the view that the Principle is *supported* by fermionic statistics, at least, since the asymmetry of the wave function ensures that no two fermions can have the same intrinsic, or state-independent, properties, nor the same state-dependent properties expressed by expectation values of all quantum-mechanical physical magnitudes.[540] The counterpart of this claim is that bosons definitely *violate*PII, since the corresponding symmetry of the wave function means that two, or more, bosons can possess all the same state-dependent properties as well as being indistinguishable.[541] A similar kind of distinction between fermions and bosons can also be found in more recent analyses, such as those of van Fraassen,[542] albeit in different terms. However, as we shall now explain, the two kinds of particle can be treated as on a par.[543]

Let us begin by noting that if the particles are regarded as non-individuals, then the question whether PII is violated is simply moot. Nevertheless, an argument can be given that even within this metaphysical framework a place can be found for the Principle, but as applied not to the particles but rather the arrangements of particles distributed over states. We recall that if we have two particles, labelled 1 and 2, distributed over two states, $|a^1\rangle$ and $|a^2\rangle$, we obtain four possibilities:

(1) Both particles in state $|a^1\rangle$
(2) Both particles in state $|a^2\rangle$
(3) Particle 1 in state $|a^1\rangle$, particle 2 in state $|a^2\rangle$
(4) Particle 1 in state $|a^2\rangle$, particle 2 in state $|a^1\rangle$

[540] For a more recent expression of this tendency, see Shadmi 1978.

[541] See Barnette 1978; Cortes 1976; and Teller 1983.

[542] van Fraassen 1989 and Saunders preprint and 2003.

[543] Our discussion essentially expands upon that of French and Redhead 1988.

We also recall that when it comes to assigning statistical weights, (3) and (4) are counted as one and the same arrangement. Now, focusing on the arrangements themselves, rather than the particles, it could be argued that this 'counting as one and the same' implies that these two arrangements are being taken as not merely indistinguishable but *identical*. (And if the two particles are individuals, then they cannot be identical and the statistical count would be incorrect; hence the particles cannot be individuals.) This identification of (3) and (4) has been adduced as grounds for holding that PII is at least supported by the way states of affairs—as represented by these arrangements—-are treated in quantum mechanics.[544]

However, this argument is mistaken. We need to bear in mind that the states we are actually concerned with in quantum statistical mechanics are:

(5) $|a^1\rangle \otimes |a^1\rangle$

(6) $|a^2\rangle \otimes |a^2\rangle$

(7) $\frac{1}{\sqrt{2}}(|a^1\rangle \otimes |a^2\rangle + |a^2\rangle \otimes |a^1\rangle)$

(8) $\frac{1}{\sqrt{2}}(|a^1\rangle \otimes |a^2\rangle - |a^2\rangle \otimes |a^1\rangle)$

These four states are mutually orthogonal and span the same subspace as those given by (1), (2), (3) and (4) but, of course, they are chosen so that (5), (6) and (7) are symmetric under particle permutations, whilst (8) is anti-symmetric.[545] The important point is that (7) is no more identical with (8) than (3) is with (4). The reason (8) doesn't feature in the counting for Bose-Einstein statistics is because it is not symmetrical; likewise, (5), (6) and (7) are eliminated for fermions. In other words, certain arrangements are eliminated not because they are identified with each other, but for reasons to do with their symmetry characteristics. And one way of understanding this, in the context of taking the particles to be individuals, is to view the states corresponding to these arrangements as not accessible to the joint quantum system, rather than as simply non-existing.[546] In other words, in terms of this package—which

[544] See Barbour 1980 and Lucas 1984, p. 131.

[545] (5) and (6) are of course the same states as (1) and (2).

[546] If we recall the view touched on earlier, that there is some justification for regarding the particles as individuals drawn from the experimental context, it is important to note that the claim is not that the system somehow evolves from (8), say, to (5) or (6) when the particle hits a scintillation screen, say; that would involve some kind of non-symmetric Hamiltonian or the measurement of a non-symmetric observable. Rather, the idea is that experimental considerations give us grounds for regarding the particles as individuals but we are unable to associate these individuals with the separate states 'entangled' in the superposition represented by (8). It is precisely this which lies behind our consideration of the impact of quantum statistics on PII.

is the appropriate one for consideration of PII—-the way quantum statistics treats the above arrangements should not be taken to support the Principle.

Turning our attention now to the particles themselves, we recall the various forms PII can take: in terms of second-order logic with equality, it can be written as

$$\forall F(F(a) \leftrightarrow F(b)) \rightarrow a = b, \tag{4.2.1}$$

where $a$ and $b$ are individual constants designating the entities concerned and $F$ is a variable ranging over the possible attributes of these entities. As we saw in Chapter 2, different forms of PII then arise depending on what sort of attributes feature in the range of $F$: PII(1) states that it is not possible for two individuals to possess all properties and relations in common; PII(2) excludes properties and relations which can be described as spatio-temporal; while the strongest form, PII(3), includes only monadic, non-relational properties. Both PII(1) and PII(2) allow for the possibility that relations might be capable of distinguishing entities and hence confer individuality.[547] However, such a possibility has been vigorously disputed on the grounds that since relations presuppose numerical diversity, they cannot account for it.[548] We shall return to this possibility in the quantum context shortly.

When it comes to the status of PII in quantum physics, we must first face a couple of broadly interpretational issues. Let us begin with PII(1). As we saw in Chapter 2, this, the weakest form of the Principle, is satisfied in the classical context because the dynamical state-description yields definite, uniquely determined spatio-temporal trajectories for each particle in the assembly. This description implicitly invokes the Impenetrability Assumption (IA) and hence distinguishability via spatio-temporal location is assured. Now, one might be tempted to conclude that PII is false for quantum particles, on the basis of the following argument: as is well known, on the standard interpretation of quantum mechanics, unique spatio-temporal trajectories do not in general exist and, in particular, (IA) is not implied by the relevant dynamical equation, namely Schrödinger's equation. If it could be shown that no other distinguishing (and hence individuating) properties could be appealed to in the quantum context, then it might be concluded that PII(1) is not satisfied.[549] Since this

[547] See for example Casullo 1984, who argues that the view of entities as nothing more than bundles of properties and relations is only plausible if based on PII(1) with relations given the capacity to individuate.

[548] See Russell 1956 and Armstrong 1978, pp. 94–5.

[549] See Belousek 1999, Ch. 5, p. 73.

is the weakest form of the Principle, PII in general would then be ruled out. However, this is clearly too quick.

First of all, one could respond by arguing that to be an individual is to be understood in terms of what we have called Space-Time Individuality, and insisting on the metaphysical importance of IA. In that case, if (IA) does not hold, then the entities concerned are simply not individuals and PII is not violated, rather it simply fails to apply. This would amount to a further version of the Received View. Alternatively one could argue that an extended form of (IA) does in fact hold in this context. Thus one way of understanding the Exclusion Principle is that it determines that no two fermions can exist in the same one-particle state, or that quantum states in which two fermions possess identical sets of one-particle quantum numbers cannot exist. Expressed in this manner, the Exclusion Principle might be regarded as a generalization of the IA in the sense that it applies not simply to the spatio-temporal coordinates of the particles, as in the classical case, but to the one-particle quantum numbers.[550] It can then be argued that in this sense, two fermions cannot be in the same state and hence IA, and by implication, PII(1)—or a suitable 'quantum' reformulation which parallels the generalization of IA—is satisfied.[551] Bosons, on the other hand, not being subject to the Exclusion Principle, would violate PII on this view and, indeed, much of the discussion of the status of the Principle in this context has focused on bosonic states, as we shall see.

However, this attempt to save PII for fermions is unpersuasive. Consider the representation of the allowed anti-symmetrical state given by (8) above, which formally incorporates the Exclusion Principle via the anti-symmetrization of the appropriate ket. It is not true that in such a state each particle is present in a different state. *Each* particle clearly 'partakes' of *both* the states $|a^1\rangle$ and $|a^2\rangle$ in the superposition of product states expressed in (8). So from this alternative perspective, it can be argued that both particles in the allowed state also have the same state-dependent properties and PII is violated. This is the perspective we shall now adopt.

The second issue concerns the attribution of such state-dependent properties in the case of 'entangled' states such as (8). In classical physics the state-dependent properties of a particle are completely specified by the maximally specific state description, as given by the location of the system in

[550] See, e.g., Shadmi 1978, p. 847.

[551] Shadmi ibid.; later we shall also consider an alternative generalization of IA in the context of the 'topological' approach to quantum statistics.

phase space. Hence the question 'Do classical particles have the same state-dependent properties?' is equivalent to the question 'Do the two particles have the same maximally specific state description?'. Unfortunately this move is not so straightforward in quantum mechanics, since here pure states play the role of maximally specific states but for 'entangled' states such as (7) and (8) there *are* no pure states which can be ascribed to the separate particles.[552] So if we were to follow the classical move and identify the relevant properties of the particles with the pure states they are in, we would be forced to conclude that there is simply no answer to the question 'Do they have the same properties?'[553]

However, there is an alternative, which is to identify these properties with the relevant 'mixed' states of the particles;[554] that is, equiprobable mixtures of $|a^1\rangle$ and $|a^2\rangle$ .This then allows us to pose the question, 'Do the particles have the same properties?' and what underpins this move is the fact that pure states and mixed states cannot be distinguished by means of observations made on one of the particles alone (we shall return to this point). The expectation value for any observable $O$ in a pure state contains, in general, 'interference' or 'cross product' terms which do not occur in the expression for the expectation value of $O$ in the mixed state. However, these terms vanish when $O$ has the form $Q \otimes I$, where $I$ is the identity operator and $Q$ is any observable referring to one particle only.[555] In general, it can be shown that pure states behave like mixtures with respect to measurements made on one particle only.[556] As we shall see, this allows us to capture the intuition that in a superposition such as (7) or (8), each particle 'partakes' of both the states $|a^1\rangle$ and $|a^2\rangle$ , in a way which violates PII.

So, let $Q$ denote a possible observable (self-adjoint operator) on a single particle. Considered as possible physical magnitudes for the joint system, there are two possibilities: $Q \otimes I$ for particle 1 having the property $Q$ and $I \otimes Q$ for particle 2 having this property. Let us denote $Q \otimes I$ by $Q_1$ and $I \otimes Q$ by $Q_2$. Interpreted as a restriction on observables, IP now says that although $Q_1$ and $Q_2$ are self-adjoint operators on the Hilbert space for the joint system, they cannot actually be observed. The intuition behind this is that observing

[552] If there were, the state of the combined system would be the tensor product of the states in question, but (7) and (8) do not have the form of tensor products—they represent superpositions of such products.

[553] As we shall see, this is precisely the line adopted in a criticism of the analysis we are about to present.

[554] See D'Espagnat 1976, pp. 58–61; he calls these states 'improper' mixtures.

[555] Observables such as this, which are unable to distinguish pure and mixed states, are called 'insensitive'.

[556] See the comment in Jammer 1974, p. 479, fn. 14.

$\mathcal{Q}_1$ or $\mathcal{Q}_2$ would involve knowing empirically which particle was which, and this is impossible if the particles are indistinguishable in the 'strong' sense. However, if our analysis of PII were to include only attributes that could be observed, we could only consider symmetric combinations such as $\mathcal{Q}_1 + \mathcal{Q}_2$ and this would be unduly restrictive. If we are to consider the *ontological* significance of the Principle, we need to discuss whether particles 1 and 2 have the same physical attributes expressed by $\mathcal{Q}_1$ and $\mathcal{Q}_2$ and their associated 'actualization' probabilities, while recognizing that these attributes can never be observed.[557] And under the 'standard' interpretation of quantum mechanics, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ must be interpreted not as possessed values but as propensities to yield specified 'actualization' results in accordance with the usual statistical algorithm for computing the associated probabilities (we shall briefly consider the implications of adopting alternative interpretations below).[558]

Following French and Redhead, we can now show that fermions violate PII just as much as bosons. With the state-dependent properties of the particles formulated in terms of physical magnitudes $\mathcal{Q}_1$ and $\mathcal{Q}_2$ pertaining to each particle separately and the superposition given in (8) represented by $|\psi\rangle$, the relevant monadic properties will be represented by $Prob^{|\psi\rangle}$ $(\mathcal{Q}_1 = q^\alpha)$ for particle 1 and $Prob^{|\psi\rangle}$ $(\mathcal{Q}_2 = q^\alpha)$ for particle 2, where the notation indicates the probability in the state $|\psi\rangle$ that the physical magnitude pertaining to either particle actualizes with the indicated value. The relevant relational properties will be represented by $Prob^{|\psi\rangle}$ $(\mathcal{Q}_1 = q^\alpha/\mathcal{Q}_2 = q^\beta)$, which refer to the conditional probabilities of actualizing one magnitude given the actualization result for the other. The values of these properties can then be calculated—and hence compared—from the joint distribution as follows:

$$\begin{aligned}
&\mathrm{Prob}^{|\psi\rangle}(\mathcal{Q}_1 = q^\alpha \,\&\, \mathcal{Q}_2 = q^\beta) \\
&= |\langle q^\alpha | \langle q^\beta | (|\psi\rangle)|^2 \\
&= \tfrac{1}{2}|\langle q^\alpha | a^1\rangle|^2 \cdot |\langle q^\alpha | a^2\rangle|^2 + \tfrac{1}{2}|\langle q^\alpha | a^2\rangle|^2 \cdot |\langle q^\alpha | a^1\rangle|^2 \\
&\quad - \mathrm{Re}\langle a^1 | q^\alpha\rangle\langle q^\alpha | a^2\rangle\langle a^2 | q^\alpha\rangle\langle q^\alpha | a^1\rangle
\end{aligned} \tag{4.2.2}$$

[557] French and Redhead op. cit., p. 239.

[558] They are referred to as 'actualization results' rather than 'measurement results' in order to emphasize the point that they are not observable (French and Redhead op. cit., p. 239, fn. 8). Although they are produced by measurement interactions, they are not produced in a way which makes them identifiable as contrasted with their particle label permuted variants (ibid.).

Summing this result over α and β to obtain the marginal probabilities and remembering that $\sum_n |q^n\rangle\langle q^n| = 1$, $\langle a^1|a^2\rangle = 0$ and $\langle a^1|a^1\rangle = \langle a^2|a^2\rangle = 1$, we get

$$\text{Prob}^{|\psi\rangle}(Q_1 = q^\alpha) = \text{Prob}^{|\psi\rangle}(Q_2 = q^\alpha) = \frac{1}{2}|\langle q^\alpha a^1\rangle|^2 \cdot |\langle q^\alpha a^2\rangle|^2. \tag{4.2.3}$$

Similarly, we obtain

$$\begin{aligned}\text{Prob}^{|\psi\rangle}(Q_1 = q^\alpha / Q_2 = q^\beta) &= \text{Prob}^{|\psi\rangle}(Q_2 = q^\alpha / Q_1 = q^\beta) \\ &= \{|\langle q^\alpha|a^1\rangle|^2 \cdot |\langle q^\beta|a^2\rangle|^2 + |\langle q^\alpha|a^2\rangle|^2 \cdot |\langle q^\beta|a^1\rangle|^2 \\ -2\text{Re}\langle a^1|q^\alpha\rangle\langle q^\alpha|a^2\rangle\langle a^2|q^\beta\rangle\langle q^\beta|a^1\rangle\} &/ \{|\langle q^\beta|a^1\rangle|^2 + |\langle q^\beta|a^2\rangle|^2\}\end{aligned} \tag{4.2.4}$$

Recalling the above identification of the monadic and relational properties of the particles in terms of the relevant probabilities, (4.2.3) and (4.2.4) show that two fermions in the entangled state given by (8) actually do have the same monadic and relational properties one to another.[559] This means that PII, is violated, even for fermions.[560]

Before we turn to the boson case, there are two further comments to make.[561] First of all, if the self-identity of a particle is represented as a relational property of the particle to itself then it can be expressed by comparing $\text{Prob}^{|\psi\rangle}(Q_1 = q^\alpha / Q_2 = q^\beta)$ as given by (4.2.4) above with

$$\text{Prob}^{|\psi\rangle}(Q_2 = q^\alpha / Q_2 = q^\beta) = \delta_{\alpha\beta} \tag{4.2.5}$$

PII would only be vindicated, however, if (4.2.5) is regarded as a monadic property of particle 2, whereas in fact it is a relational property of particle 2 to itself, which is also true as a relation of particle 1 to *itself*.

[559] Butterfield has extended this result to include relational properties which involve a third component, where these properties are identified with the conditional probability for a quantity on this third component to have a given value, conditional on a quantity on the first/second component having a value (Butterfield 1993, p. 464). Huggett has recently gone even further and has shown that the failure of PII for a pair of particles is equivalent to their (anti-)symmetrization and not merely implied by it (Huggett 2003).

[560] It is worth noting that these relational properties expressed by the conditional probabilities do not supervene upon the monadic properties expressed by the marginal distributions. Teller (1986) refers to these as 'non-supervenient' or 'inherent' relations, and we shall return to consider them below. Of course, a supporter of PII could insist that such relations lie beyond the scope of the Principle (perhaps citing Leibniz's claim that all relations are reducible to monadic properties), but given the ubiquity of such relations in quantum mechanics, this would have the effect of simply rendering the Principle irrelevant in the context of modern physics.

[561] See French and Redhead op. cit.

Secondly, if we substitute $\alpha = \beta$ in (4.2.4) above, then we obtain $\text{Prob}^{|\psi\rangle}(Q_1 = q^{\alpha}/Q_2 = q^{\beta}) = 0$. Putting this into words: if actualization of $Q_2$ gives a certain value then there is zero probability that a concurrent actualization of $Q_1$ will yield the same value. This is the true meaning of Pauli's Exclusion Principle, but it has nothing to do with PII, as long as we continue to adopt the standard view that actualization results do not reveal pre-existing values.[562] Of course, if this standard assumption were to be dropped, then the considerations just given could be used to argue that PII is vindicated for fermions. We shall consider two ways in which this assumption can be dropped shortly, together with the consequent impact upon PII.

Let us now consider bosons. This is the case most often discussed in the literature and it is typically assumed that the purported violation of PII depends on consideration of states such as (5) or (6), where both particles can indeed be attributed the same pure state. Thus Cortes, for example,[563] considers the example of two photons in a mirror-lined box, which are indistinguishable and also, he argues, are spatio-temporally related to every other thing in the universe in exactly the same way. Furthermore, their spatio-temporal trajectories or histories cannot serve to individuate them, since at the point at which these histories cross, and the photons share all spatio-temporal properties in common, it is logically impossible to determine which photon at that point has which history.[564] Nevertheless, that there are two photons in the box can be determined by noting, for example, that the total energy of frequency $\nu$ in the box is $2h\nu$, and then employing Planck's equation. Hence PII is violated. Clearly the relevant principle that is being adopted here is Space-Time Individuality and Cortes' point can be viewed as a perfectly general one to do with the failure of spatio-temporal trajectories to individuate in situations where the Impenetrability Assumption fails to hold.

However, Barnette has rejected Cortes' claim on the grounds that it confuses epistemological issues with metaphysical ones:[565] the fact that, after the trajectories cross, there is no way of *telling* which photon came from where, does not compel us to give up the metaphysical claim that the predicate 'possessing history *H*', say, is satisfied by one and only one of the photons for all times after the crossing. Given that this predicate is satisfied by one and only one of the objects for all times, it follows that for all times, including those after the crossing, there is some description which holds for one object and

[562] French and Redhead op. cit.

[563] Cortes 1976.

[564] Ibid., p. 503.

[565] Barnette 1978.

not the other. Hence PII is not violated. The point Barnette is making is akin to that underpinning the 'Scholastic' distinction between distinguishability and individuality: just because there is no predicate which satisfies the former, does not mean that the latter is compromised.

This sort of response is problematic, however, when it comes to the status of PII. Teller has objected that it is not the case that in *practice* we cannot distinguish the two photons, but, rather, that there is nothing in *principle* which can serve to individuate them.[566] In such situations, epistemological issues cannot be so cleanly separated from metaphysical ones and in this case, the relevant metaphysical issues may depend on what characteristics are available which could, in principle, serve to individuate the particles. Given the absence of such characteristics for the two photons in the same state, it is begging the question to say that they are individuated by differences in some historical property. Furthermore, it is not clear how we should understand the notion of 'having a history' in this context: if to 'have a history' involves not just travelling along some spatio-temporal trajectory but also satisfying the Impenetrability Assumption then on the most obvious construal of the latter in the quantum context, photons do not have 'histories'. We shall return to this issue as it crops up in other analyses, as we shall see.

There is a related concern about the legitimacy of talk of continuous 'histories' when we shift to the QFT perspective. As Ginsberg (1981) has pointed out, the introduction of creation and annihilation operators can be interpreted as breaking up the history. What we have is not bosonic trajectories crossing, or merging, and then separating but rather a situation where the field is initially in a two-particle state, with one photon in mode *A* and one in mode *B*, say, followed by a two-particle state with both photons in the same mode, *P*, say, and then further followed by a two-particle state with one photon in mode *C* and one in mode *D*. From this perspective, we have a sequence of creation and annihilation of photons—understood as field excitations—in these various modes and, according to Ginsberg, it is simply false to claim that the predicate 'object identical to the object having history *A*' is possessed by either photon in mode *P* since neither of these is identical to any of the photons in modes *A,B, C* or *D*. Furthermore, given that the two photons existing in mode *P* are not identical to the ones that exist before or after, and are excitations of the same field mode and possess all properties in common, yet are numerically distinct, PII is violated.[567]

[566] Teller 1983, p. 314.

[567] Ginsberg 1981, p. 491.

However, as the discerning reader may have noticed, this conclusion appears to have been reached by blurring the distinction between the description of the photons as particles, or objects in general, and the description of them as field excitations. PII can only be taken as violated if the photons are considered to be numerically distinct. They are if the particle interpretation is adopted according to which the total energy $2h\nu$ is regarded as due to two photons each having energy $h\nu$. However, in the field representation, what we have in this case is simply the excitation level of a field oscillator being raised by two units. From this perspective, we do not have two indiscernible individuals and the issue as to the status of PII is again obviated. In other words, the question whether or not PII is violated depends on which perspective of the two-photon state one adopts.[568] If the view of photons as merely field excitations is understood as the QFT counterpart of our talk of particles as non-individuals, then what we have here is an instantiation of the metaphysical underdetermination between individuality and non-individuality.[569] Of course, if we were to adopt the package which regards the photons as particles—subject to creation and annihilation (which may be problematic)—then Ginsberg's conclusion stands.

Returning to our analysis, if the bosonic state (5)—namely, $|a_1\rangle \otimes |a_1\rangle$ —is denoted by $|\Phi\rangle$ , then one obtains the following results, corresponding to (4.2.3) and (4.2.4) above:

$$\text{Prob}^{|\Phi\rangle}(Q_2 = q^{\alpha}) = |\langle q^{\alpha}|a^2\rangle|^2 \tag{4.2.6}$$

and

$$\begin{aligned}\text{Prob}^{|\Phi\rangle}(Q_1 = q^{\alpha} / Q_2 = q^{\beta}) &= \text{Prob}^{|\Phi\rangle}(Q_2 = q^{\alpha} / Q_1 = q^{\beta}) \\ &= |\langle q^{\beta}|a^2\rangle|^2\end{aligned} \tag{4.2.7}$$

Again, the two particles possess all monadic and relational properties in common and PII is violated. However, it is important to note that this conclusion also holds for the state given by (7) ($\frac{1}{\sqrt{2}}(|a^1\rangle \otimes |a^2\rangle + |a^2\rangle \otimes |a^1\rangle)$), where two different states are involved. In this case the results (4.2.3) and (4.2.4) apply, with the minus sign in front of the 'interference' term in (4.2.4)

[568] Teller op. cit.

[569] And it is in his discussion of PII that Teller repeats Hesse's analogy of money in a bank account.

replaced by the plus sign. The point is, from the point of view adopted here, we don't have to treat bosons any differently from fermions and the discussion of the nature of two-photon states is completely undercut.

Do similar results hold for paraparticles? In this case, there do exist states for which the monadic properties of the particles are not the same. However, it can be shown that there are possible paraparticle states for which PII is violated in the same way as for bosons and fermions. We shall not go through all the details here[570] but basically one can show that there exist states for which particles 1 and 3 (recall that parastatistics only deviate from 'normal' quantum statistics for greater than two particles) have the same monadic and relational properties expressed by the relevant distributions, as above, but that these differ from the properties of particle 2. Hence PII is violated for paraparticles 1 and 3 in such states.[571]

The upshot, then, is that if the non-intrinsic, state-dependent properties are identified with all the monadic or relational properties which can be expressed in terms of physical magnitudes associated with self-adjoint operators that can be defined for the particles, then it can be shown that two bosons or two fermions or two paraparticles in a joint symmetric or anti-symmetric state respectively have the same monadic properties and the same relational properties one to another. Given this identification, even the weakest form of the Principle, PII(1), fails and the Principle of Identity of Indiscernibles is straightforwardly false.[572] Hence it cannot be used to guarantee individuation via the state-dependent properties and we need to shift to some other approach.

Before we move on, however, we must address two sets of challenges to this analysis. The first, like the analysis itself, retains the more or less standard interpretative framework for quantum mechanics but presses for alternative understandings of certain crucial elements of this framework. The second kind of challenge urges us to step outside of this framework altogether and offers alternative perspectives on the status of PII from these non-standard interpretative positions.

Now, there are a number of ways in which the analysis might be resisted while remaining within the standard interpretation. One might, for example, question the use of improper mixtures to attribute separate states and hence properties to the particles. Let us recall the general form of the argument

[570] See French and Redhead op. cit., pp. 242–4.

[571] For a more general and more detailed discussion, see Huggett 2003.

[572] For further discussion see French 1989b.

in the case of fermions: given a joint state for the assembly such as (8), the state that can be attributed to each fermion is the same (improper) mixture. It is this attribution which generates the same expectation values for each of their respective observables and leads to the problems for PII.[573] However, one can question whether two separate states can be attributed to the fermions in the first place.[574] If the relevant states are pure states, then such attribution is unproblematic of course, but it is precisely because of the 'entangled' nature of states such as (8) that we cannot attribute separate pure states to the particles. Hence we must employ mixed states as a kind of stand-in. We recall our justification above for doing this and representing the relevant properties of the particles in terms of their mixed states, which hinges crucially on the point that pure states and mixed states cannot be distinguished by means of observations made on one of the particles alone. However, the criticism continues, the fact that epistemologically speaking we cannot tell the difference between mixtures and pure states should not be taken to imply that *ontologically* we can talk of the particles as possessing separate states.[575] And it is the ontology that is precisely at issue here.

Massimi has pressed this line of attack by inviting us to consider the Eigenstate-Eigenvalue link. This dictates that a system can be said to have a certain property if and only if it is in the corresponding eigenstate of the observable concerned. In other words, a system $S$ possesses a monadic state-dependent property $Q$ in state $\Phi$ iff:

A. $\text{Prob}^{|\Phi\rangle}(Q_S = q^{\alpha}) = TrP_Q(q^{\alpha}).\ P^{|\Phi\rangle} = 1$; and
B. $\Phi$ is a pure state for $S$.

It then follows that systems in entangled states, such as (8), do not possess their own definite states, nor, consequently, do they possess their own definite monadic properties. Furthermore, the mixtures themselves violate the following criterion for 'ontological state-separability' and hence cannot act as separate states:

> Given any composite system, the quantum states of the subsystems are ontologically separate … iff 1. each subsystem has definite (though possibly unknown) values for a complete set of compatible observables pertaining to that subsystem alone 2.

[573] See also Margenau 1944.

[574] Massimi 2001; see also Mittelstaedt and Castellani 2000.

[575] Massimi op. cit.

the afore-mentioned ontologically separate states of the subsystems determine wholly their joint state.[576]

Improper mixtures violate both conditions by their very nature. Hence the states are not ontologically separate and consequently, one cannot attribute monadic state-dependent properties to each fermion. In this case, then, there is no relevant context for asking whether the two particles are indiscernible to begin with and PII does not even apply.[577]

There are two immediate comments to make regarding this result. The first is that in terms of our discussion, Massimi's conclusion amounts to the claim that the fermions cannot be considered to be individuals to begin with. Now, one way of responding to this is to reject the criterion for ontological state-separability as the basis for attributing individuality to the particles, and appeal to other grounds, namely that we appear to observe (the traces of) distinct particles via scintillation screens etc.; this is what we have identified as Bohr's 'experimental' individuality or what Toraldo di Francia calls 'pseudo-individuality'. On this basis one starts with the presupposition that the particles are individuals and then looks for the appropriate representation of the relevant properties, in order to see whether PII holds or not (pure states are not available, hence we invoke mixtures and the analysis proceeds).

Massimi's response is straightforward: take the contrapositive of PII—if *a* and *b* are numerically distinct individuals, then there exists at least one property that *a* has and *b* lacks or vice versa. Then the antecedent is satisfied, but still, she insists, the consequent is problematic because the relevant state-dependent properties are simply not possessed by the fermions. Hence PII fails, not because it is false but because the conditions are not met for it to apply. At this point the argument seems to boil down to the issue of what it means to say that the particles possess state-dependent properties. Given that we are dealing with an entangled state, the best that we can do is to use improper mixtures as a kind of substitute for pure states, and use them to represent the possession of state-dependent properties.[578] Epistemically we are justified in making this move. Massimi agrees that this is the best we can do but insists that it is still not good enough—ontologically, the move is problematic. Here again we touch on a tension between epistemology and ontology—one that

[576] Massimi ibid., pp. 320–1. As Massimi notes, condition 2 amounts to what is known—in the context of the Bell and EPR results—as 'factorizability': the joint probability of the expectation values relative to the two subsystems is the product of the probabilities bearing respectively on each subsystem alone. We shall return to this condition below.

[577] Cf. French 1985, p. 203.

[578] See, again, French and Redhead op. cit.

becomes particularly acute for those anti-substantivalist upholders of PII who typically want to defend and deploy the Principle on epistemological grounds (we recall Leibniz's infamous example of searching for indiscernible leaves in the Herrenhausen gardens!) but who now find themselves having to appeal to Massimi's ontological criterion in order to escape the conclusion of the above analysis.

The second comment blunts the force of Massimi's conclusion: her argument applies to *monadic*, state-dependent properties only, as she herself acknowledges.[579] It is crucially based on the claim that improper mixtures do not represent ontologically separate states nor do they encode *genuinely* monadic properties. Owing to their improper and 'holistic' nature, what they actually encode are *relational* properties, and it is precisely such properties that the analysis is concerned with. What this means, of course, is that it still goes through for PII(2), and hence we can still say that in these terms, PII is violated, but not for PII(3). In effect, as Massimi acknowledges, what she has shown is that in the quantum context those versions of PII which allow relations to individuate are not the *weakest* forms of the Principle, but the *only* forms which are applicable. In the context of this ontological framework, what this weakened conclusion suggests is that we must allow for the possibility of the individuality of the particles to be grounded in relational terms. Of course, the analysis shows that in these precise terms PII fails but, as we shall see, an alternative has been proposed which can allow for this possibility.

Before we consider this option, however, we need to consider another line of attack which questions the very characterization of properties in probabilistic terms. Belousek suggests that this characterization presupposes a "… questionable interpretation of particle properties for which [French and Redhead] provide no compelling motivation".[580] His specific concern is with the identification of probabilities as predicates 'in their own right'; that is, the characterization presupposes a propensity interpretation of quantum probabilities which can be plausibly denied. As an alternative he offers the Eigenstate-Eigenvalue link which implies, we recall, that in a state such as (8), for which all the single-particle states will in general be represented by mixed states rather than pure states, no definite values can be assigned to any single-particle observables and hence no state-dependent properties can be attributed to the particles.[581] He then concludes that, on this basis, PII(2) and

[579] op. cit., pp. 326–7.

[580] op. cit., pp. 76–8. As he notes, Butterfield (1993, pp. 461–2) simply asserts that such an interpretation is 'natural' (Belousek op. cit., pp. 77–8, n. 42).

[581] Ibid., p. 78.

PII(3) are still violated and hence PII is false in general. Unfortunately, however, this is *not* the conclusion that can be drawn; rather, from the implication of the Eigenstate-Eigenvalue rule that systems in entangled states possess neither their own definite states, nor, consequently, their own definite properties, all one can say is that PII is *inapplicable* in this case. Furthermore, this helps explain why French and Redhead adopt the interpretation they do—in order to determine whether PII is actually false or not we need to set the situation up in such a way that the Principle can actually be applicable in the first place.[582]

Let us move on to consider the second set of challenges which suggest one should shift outside the orthodox framework entirely. Of course this move is typically made as a response to interpretational problems in general but if it can save PII as well, this will be seen by supporters of the Principle as a bonus.

Perhaps the best-known proponent of this kind of move is van Fraassen, who has explored the possibility of saving PII through his version of the 'modal' interpretation of quantum mechanics.[583] The central idea is to distinguish between two kinds of state: the 'value' state, which is specified by stating which observables have values and what they are; and the 'dynamic' state, which is specified by stating how the system will develop both if isolated and if acted upon in some definite fashion.[584] The evolution of dynamic states is deterministic, in accordance with Schrödinger's equation, but the value state changes unpredictably, within the limits set by the dynamic state.[585] As is well known, this provides a solution of the measurement problem: the outcome of the measurement of some observable gives us an indication of the value state, but we cannot infer what dynamic state the system is in (thus the Eigenstate-Eigenvalue link is broken). In particular, there is no collapse of the dynamic state, but rather an indeterministic 'jump' from one value state to another.

The modal interpretation has been subjected to extensive criticism,[586] but it does allow us to retain PII in the following way: first of all, from what has just been said it is clear that on this interpretation the value of an observable at a particular time cannot be used to directly predict its value at some future time. Hence actual values of observables are not empirically significant in the sense

[582] Actually, when the French and Redhead paper was written, the interpretation of physical properties as propensities to yield actualization results, rather than possessed values, was seen as a straightforward component of the 'orthodox' view of quantum mechanics. Belousek thinks the orthodox interpretation is fatally flawed in various ways, but of course French and Redhead might well insist that it is surely a legitimate exercise to discover whether PII is violated within such an interpretation before moving on to the alternatives!

[583] van Fraassen 1991.

[584] Ibid., p. 275.

[585] Ibid., p. 277.

[586] See Dieks and Vermaas 1998.

that … "they do not increase predictive power if added to a description of the concurrent dynamic state";[587] in van Fraassen's terminology they are 'empirically superfluous' and correspond to 'extra-dynamical unused structure'. Now, in the case of fermions, the dynamic state of the aggregate will be anti-symmetric, of course (for an aggregate of two fermions it will be (8) above) and the dynamic states of each sub-aggregate (that is, each fermion) will be the same, but it is possible to assign distinct *value* states to each particle.[588] Thus his crucial distinction allows van Fraassen to say that although the dynamic states are assigned exactly as in the French-Redhead argument above (following Margenau), PII is not violated since the *value* states are different.

Now one can immediately raise the concern that it seems odd for an empiricist, such as van Fraassen, to be preserving PII—which was supposed to ground individuality without appealing to some form of Lockean substance—by invoking *empirically superfluous* factors as embodied in these value states! This concern meshes with a general criticism of the division between dynamic and value states[589] which has been applied to the present context in the following way: if the two fermions are going to be individuated via PII, then the relevant properties represented by the value state attribution must be 'objective', in the sense of 'corresponding to really distinct physical situations'.[590] However, given an assembly, the choice of what constitutes a sub-system of that assembly is arbitrary and hence the state-dependent properties that can be assigned to each sub-system will be different, depending on that choice. Hence the state-dependent properties assigned to the subsystems—as embodied in the value state attribution—are not objective in the above sense. Rather, they seem to have a 'merely conceptual nature'[591] and as such they cannot support the preservation of PII.

Of course the issue is not straightforward, depending as it does on a distinction between 'objective' and 'merely conceptual' properties, but the concern is fundamental: what properties are we allowed to include within the scope of *F* in equation (2) above? Following the above point, at least some proposals for such 'properties' seem to slip beyond the pale. Thus, in a work explicitly based on Mirman's attempt to analyse the 'experimental meaning' of individuality,[592] de Muynck has pressed the view that the particle labels themselves should be regarded as intrinsic properties of the particles.[593] This suggests a rather bizarre

[587] Ibid., p. 277.

[588] Ibid., pp. 423–9.

[589] Ghirardi 1997.

[590] Massimi op. cit., p. 318, fn. 11.

[591] Massimi ibid.

[592] Mirman 1973.

[593] de Muynck 1975, p. 327.

metaphysics of property, as particle labels are not the subject of any theory, nor are they invoked to account for the behaviour of the particles. Regarding them as 'extra-dynamical' suggests an analogy with, say, the colour of a billiard ball, at least as far as collision phenomena are concerned.[594] But this is a misleading analogy: the colour of a billiard ball can be regarded as a secondary property related, by some body of theory, to the primary properties of the fundamental particles of which the ball is composed. These latter are not taken to possess such secondary properties and it is difficult to see what meaning could be given to a primary property that is both intrinsic and yet unconnected to the dynamical behaviour of the particles. The claim that such labels can generate the qualitative difference necessary to preserve PII is simply not plausible. This is an extreme example but it serves to illustrate the point that although we can always find a way of saving PII, the metaphysical price may seem exorbitant.

Now, what about bosons? In van Fraassen's scheme, two bosons would have the same dynamic state and the same value state[595] and PII would appear to be violated. At this point, acknowledging that he is stymied, van Fraassen makes a familiar move and appeals to the spatio-temporal trajectories of the particles.[596] Each boson is individuated by its history, where this historical individuation is to be understood as 'empirically superfluous', as far as the statistics are concerned.[597] Of course, van Fraassen is aware that more than one boson may occupy the same state, leading to the (quantum mechanical equivalent of the) failure of Impenetrability, as we noted above. However, he recalls Reichenbach's point that one can either regard this as a failure of trans-temporal identification or as the manifestation of causal anomalies since it amounts to correlation without common cause.[598] Since van Fraassen accepts that any account which embodies the principle of common cause conflicts with the empirical predictions of quantum mechanics,[599] he is perfectly happy

[594] Sudarshan and Mehra 1970, p. 245.

[595] Ibid., p. 429.

[596] As he notes (ibid., p. 432), he suggested this move in a paper presented to the American Philosophical Association in 1969 and Cortes' work can be seen as responding to it.

[597] Ibid., p. 430. Interestingly, given the content of the rest of our book, van Fraassen goes on to record Mittelstaedt's emphasis on the lack of empirical significance of trans-temporal identification within quantum mechanics. However, whereas Mittelstaedt takes this to imply that we must exclude non-descriptive singular terms (such as names) from our description of the aggregate, and following him, Dalla Chiara and Toraldo di Francia introduce non-standard 'quasets' to describe such assemblies (as we shall discuss in Chapter 5), van Fraassen insists that we can effectively retain such terms—and the supporting framework of individuation—but understand them as empirically superfluous (ibid.). In effect, this difference in attitudes represents the kind of metaphysical underdetermination we are exploring in this chapter.

[598] Reichenbach 1956, section 26.

[599] On the grounds that this follows from consideration of Bell's Theorem, to be touched on below.

to take the second option, namely, keep trans-temporal identification and accept causal anomalies.[600]

Barnette's accusation against Cortes—of confusing metaphysics with epistemology—is particularly apt in this context. By insisting that bosonic histories are empirically superfluous, van Fraassen accepts that we have no epistemic access to them but the point is that metaphysically these histories do the job.[601] Teller's objection that there is no way of individuating the particles 'in principle' precisely misses this point; as we noted above, if we make the distinction between distinguishability and individuality, then whether or not we can epistemically distinguish the bosons has no bearing on whether we can metaphysically individuate them. Indeed, van Fraassen might well argue that invocations of 'in principle' impossibility obviously depend on what the principles are; if the principles are empirical then of course the particles cannot be individuated in these terms, but the notion of empirical superfluity goes beyond such principles. However, this serves to further highlight the point we already made above—that it is odd that an interpretation which originally sought to avoid such otiose conceptions as Lockean substance should find itself having to include empirically superfluous factors within the scope of the quantifier in the expression for PII.

Finally, it might also be asked why such different metaphysical schemes are required for fermions and bosons. Of course one answer is that this difference corresponds to the physical difference between these two kinds of particles, namely that fermions include all those particles that constitute *matter*, whereas the boson kind covers those particles which mediate the *forces* between bits of matter. As it stands, however, this amounts to nothing more than an assertion of parallelism. What one wants is some explanation of what the first difference has to do with the second. And given that, as far as the statistics is concerned, the difference between fermions and bosons amounts to no more than a difference of sign in expressions (7) and (8) above, coming up with such an explanation might be tricky. As we shall now see, a similar point arises with regard to another attempt to preserve, not PII itself, but what is taken to be a more appropriate substitute.

Saunders has recently suggested that PII should be replaced by a 'Principle of the Identity of Indiscriminables'.[602] Roughly speaking this is the condition

[600] op. cit., p. 432.

[601] It is important to be clear that—as van Fraassen himself says—what he is talking about through all this is *interpretation*. And for the constructive empiricist an interpretation tells us, not how the world *is*, but how the world *could be*.

[602] Saunders preprint and 2003a.

that $x = y$ (where $x$, $y$, $u_1$, $u_2$, … are variables) if and only if, for all unary predicates $A$, binary predicates $B$, …, $n$-ary predicates $P$, we have

$$A(x) \longleftrightarrow A(y)$$
$$B(x,\ u_1) \longleftrightarrow B(y,\ u_1);B(u_1,\ x) \longleftrightarrow B(u_1,\ y)$$
$$P(x,\ u_1 \ldots u_n) \longleftrightarrow B(y,\ u_1 \ldots u_n) \text{ and permutations}$$

together with all universal quantifications over the free variables $u_1$, …, $u_n$ other than $x$ and $y$.[603] If the relevant language contains monadic predicates only, then this principle amounts to the claim that two entities are identical if and only if they have all properties in common. Two entities are said to be absolutely discernible if there is a formula with only one free variable which applies to one entity but not the other. With only monadic predicates allowed, the principle states that numerically distinct entities are absolutely discernible. If relations are admitted, however, one can have entities which are not identified by the principle yet are not absolutely discernible. Two entities are said to be relatively discernible if there is a formula in two free variables which applies to them in one order only. If there is a formula that applies to them in any order, but is not satisfied by ordered pairs whose elements are the same object, then they are said to be weakly discernible. If neither absolutely nor relatively discernible, they are said to be indiscernible.[604] This yields a form of PII which Saunders suggests is in accord with modern logic: objects are numerically distinct only if absolutely, relatively or weakly discernible.[605]

This, it is claimed, is more natural from a logical point of view as it is immune to the standard counter-examples which beset PII as standardly understood. Thus the the infamous two globes, floating a certain distance apart in an otherwise empty space, are weakly indiscernible and consequently not a counter-example to this form of the Principle.[606] It is also better suited to

[603] Saunders preprint, pp. 10–11; for a more developed account see Saunders 2003a, 2006.

[604] Saunders preprint, p. 11.

[605] Of course we can have *a* distinct from *b* even if they have all their properties in common, so violating PII if we adopt Henkin semantics for second order logic, as we shall see in Chapter 6. What this demonstrates is that the validity of a logical law does not entail that the world is as the theory says it is. The metaphysical conclusions one draws depend on the structure one is working with. Identity, in particular, must be defined relative to a structure—a point we shall be returning to in subsequent chapters.

[606] See Saunders 2003a, 2006.

quantum mechanics in that, unlike standard PII, it is not violated by fermions, since an irreflexive relation always exists between them. Consider, for example, two fermions in a spherically symmetric singlet state. The fermions are not only indistinguishable, but also have exactly the same spatio-temporal properties and relations between themselves and everything else. However, each satisfies the symmetric but irreflexive relation of "having opposite direction of each component of spin to …" and so they are weakly discernible and hence not identified according to the principle.[607] Thus for fermions, at least, we have the possibility of grounding their individuality via this Principle of Identity of Indiscriminables, without having to appeal to anything like primitive thisness. No such possibility exists for bosons, however, and Saunders adopts the 'Received' option of regarding them as non-individual field quanta. He takes this metaphysical difference as tracking the physical one between the 'stable constituents of ordinary matter' (fermions) and gauge quanta (bosons), although, as we indicated above, it is not clear why the metaphysics should follow the physics in this particular way, or at all.[608]

There are three sets of concerns that arise with this approach. The first has to do with its explicitly Quinean features. Our discussion so far might be characterized as 'Scholastic' in nature, with its emphasis on the distinction between distinguishability and individuality and the priority given to metaphysics over epistemology (even van Fraassen explicitly cites Aquinas!). Saunders calls his Principle 'Quinean', since he draws it from Quine's expression of Leibniz's law.[609] The difference is crucial, given Quine's views, in general, of identity, its role in logic and its relationship with the very notion of an entity and, in particular, of objects as values of variables in a logical schema. Thus Saunders characterizes his proposal in the following terms: given a situation in which we do not know what physical objects there are but only the relevant predicates, terms and the connections between them, as represented in the appropriate theory, we should admit no more entities into our ontology than are required

[607] Saunders preprint, p. 11.

[608] Saunders' response is that he is concerned with descriptive metaphysics and thus his aim is to preserve 'a good part' of established physical practice; ibid. p. 6, fn. 7. Since ordinary objects turn out to be individuals on his account, their (stable) constituents should too, he insists. It is not clear to us what the force of the 'should' is here. Indeed, this is to deny the point made by Bohr, Toraldo di Francia and others that when we try to import the criteria of individuality appropriate for 'everyday' objects into the domain of quantum objects, we run into trouble. Of course, Saunders would deny that we do run into trouble, at least not if we adopt his approach! As we have repeatedly stated, our stance here is that one can maintain that quantum objects are individuals if one wishes, but that if one were to adopt the Received View, it should be in the context of an appropriate logico-mathematical framework.

[609] See Quine 1976.

by the distinctions that can be established using these predicates and terms.[610] In other (cruder) words, we should read our ontology off the relevant theory, and, given the existence of the appropriate predicates and relations, the identity of the objects of that ontology will be secured by the Quinean scheme above. Now, from the perspective adopted in this chapter—that it is not necessary to regard quantum particles as non-individuals—Saunders' demonstration that fermions, at least, can be regarded as (Quinean) individuals, is to be welcomed. Nevertheless, from the point of view we shall be exploring in subsequent chapters, Quine's famous catch-phrase that objects are values of variables is problematic. Variables range over sets, and sets (in the standard set theories) are understood as collections of *individuals*. Thus, from *this* perspective, the Quinean approach adopted by Saunders may appear to be begging the question.

The second concern has already been touched upon: in what sense can we say that two electrons in an entangled state *have* or *bear* the relation of "having opposite direction of each component of spin to …"? After all, it might be claimed, it is only upon measurement that we can even talk of two entities with their own separate states. Thus, to presuppose there are two electrons in the entangled state bearing these irreflexive relations to begin with appears to beg the very question at issue. However, as Saunders makes clear, he is working with a relational conception of the quantum state here and this specific irreflexive relation is simply a manifestation of the anti-symmetric state itself: since they are in such a state, the electrons must have opposite spin. Furthermore, to insist that we can only talk about two entities in such a state if they can be said to possess separable states—which they obviously cannot—is equivalent to insisting that only such states, corresponding to monadic properties, allow us to distinguish and hence individuate the entities. But now the question begging has been turned, since it is precisely this latter insistence that Saunders wants to move away from.[611] This point will become clearer after we consider the third concern.

This reprises the worry hinted at previously, regarding the individuating power of relations: doesn't the appeal to irreflexive relations in order to ground the individuality of the objects which bear such relations involve a circularity? In other words, the worry is that in order to appeal to such relations, one has already had to individuate the particles which are so related and

[610] op. cit., p. 4.

[611] This is precisely the insistence expressed in Margenau's approach and hence Massimi's rejection of it, as noted previously, meshes nicely with Saunders' argument.

the numerical diversity of the particles has been presupposed by the relation which hence cannot account for it. One possible response is to suggest, again, that this worry confuses epistemological issues with metaphysical ones. Let us grant that to *know* that a relation is irreflexive presupposes that one knows that the relata are diverse. Still this does not imply that, *ontologically* speaking, such relations cannot individuate. Of course, if one thinks that ontological issues regarding individuality depend on epistemological issues concerning individuation—which seems to mesh with the attitude behind PII, at least—then in the absence of any other individuating attributes, appealing to irreflexive relations in order to individuate the particles is still going to seem like a question-begging move.[612] If, on the other hand, one holds that such ontological issues can be addressed irrespective of the epistemological ones, then the move may seem more palatable. The problem, however, is that if one holds such a view then one is obviously going to have to give some account of the metaphysics of individuality which in effect peels it away from the epistemology of individuation and distinguishability.

We have to be a little careful here. Consider again the relation "having opposite direction of each component of spin to …". The circularity may appear to arise if the "…" in the description of the relation is taken to be a form of label which designates an object bearing such a relation. The problem then is how such an object can be designated, as an individual object, in a way that is conceptually prior to its entering into the relation, in order that it could be said to enter into the relation in the first place. Notice that this is not an issue of how we *know* there is an object there to begin with: even if we separate the metaphysical or ontological issues cleanly from the epistemological ones and make the conceptual distinction between individuality and distinguishability, the problem still arises. And if we solve it by appealing to something like primitive thisness then we lose the motivation for bothering with either PII or Saunders' alternative to begin with. Formally, of course, only bound variables need fill the "…" for the formulas to be such that the only models of this formula contain at least two elements of the domain, where each such element has opposite direction of each component of spin to the other. But still, this presupposes a domain of elements capable of 'having' such properties and the worry remains: we need to have the elements in the first place in order to talk about the relations which effectively individuate those elements.

[612] This issue is touched on in French 1989b.

The issue is one of ontological priority. One might hold the view that relata have ontological priority over relations, such that the former can be said to 'enter into' the latter. Barcan Marcus, for example, writes that, "Individuals must be there before they enter into relations …".[613] On such a view, relations, such as the irreflexive ones above, cannot serve to ground the individuality of the entities which enter into these relations. The Quinean approach which Saunders adopts masks this issue. On this approach, as we indicated, we read our ontology off the theory and the identity of these objects is then secured by the above schema. One could adopt a different understanding of quantification over the relevant variables and the way this functions as a guide to ontology. Barcan Marcus, again, insists that "… if we want to discover which objects a language or theory takes to be individuals, we look to see which objects are such that they can meaningfully enter into the identity relation".[614] The difference is that—to use slogans again—whereas Quine maintains 'No entity without identity', Barcan Marcus insists on 'No identity without entity.'

Leaving this difference aside for now, the use of irreflexive relations to ground the individuality of quantum particles via Saunders' version of PII obviously fails to mesh with the above view. As an alternative, one could adopt a metaphysics in which relations have ontological priority in some sense, so that instead of talking of relata 'entering into' relations, one talks of the intersection of relations as 'constituting' relata, in some sense. This appears similar to certain 'structuralist' proposals according to which physical objects are reconceptualized as 'nodes' in or intersections of certain structures. We have already touched on such proposals in Chapter 3. Similar views have been elaborated by present-day structuralists and Saunders' work can perhaps most appropriately be placed within this context.[615] Of course, it must be acknowledged that in order to *describe* the relation—either informally as above or set-theoretically in terms of an ordered tuple $\langle x, y \rangle$ —we have to introduce some form of label, as in the example above, but description should not dictate conceptualization. The label can be understood as a kind of place-holder which allows us to effectively talk of objects, whose individuality is then understood via this form of PII. What this amounts to, then, is a structuralist ontology which allows for individuation via relations. In effect, what Saunders has done is to preserve something akin to PII but at the cost of a shift away from the standard metaphysics of individuality and objecthood. We shall briefly discuss other

[613] Barcan Marcus 1993, p. 20.

[614] op. cit., p. 25.

[615] See, e.g., Ladyman 1998; French and Ladyman 2003; Saunders 2003b.

structuralist reactions to these sorts of issues later but for the moment let us pursue another 'classical' option—that of grounding the individuality of particles in some form of space-time trajectory.

## 4.3 SPACE-TIME INDIVIDUALITY AND CONFIGURATION SPACE

We have seen that individuating the particles in terms of their properties in general is problematic. The classical alternative of both individuating and distinguishing them in terms of spatio-temporal location, in conjunction with the Impenetrability Assumption (playing the role of guarantor) also faces acute difficulties. If the standard interpretation is adopted then it can be shown that the family of observables corresponding to the positions of single particles cannot provide distinguishing spatio-temporal trajectories. However, as is well known, the Bohm interpretation does allow for the introduction of such trajectories; indeed, the only observable admitted is that of position. Hence, this might seem the natural home for some form of Space-Time Individuality. Let us briefly consider this option.

On this interpretation we have a dual ontology of point particles plus 'pilot' wave, where the role of the latter is to determine the instantaneous velocities of the former through the so-called 'guidance equations'.[616] These are taken to 'complete' the standard formulation of quantum mechanics so that, in addition to the quantum state, whose development is determined by the Schrödinger equation, there is also a set of single-particle trajectories, each of which is determined by the guidance equation (plus the initial positions of the particles, of course).[617]

It is part of the attraction of this approach that it retains, or appears to retain, a form of classical ontology in which the standard philosophical problems of wave-particle duality are apparently resolved by effecting a metaphysical split or division of labour, with particles traversing well-defined spatio-temporal trajectories guided by the pilot wave. This obviously meshes well with our metaphysical package of regarding the particles as individuals.[618] Indeed, a form of PII(1) can now be defended against the results of the previous section. As Belousek has emphasized, the conclusion that PII is violated is based on

[616] Ibid.

[617] For a review of this interpretation see Cushing *et al.*1996.

[618] See Brown *et al.*1999.

the restriction of the relevant set of properties to those that can be represented in terms of Hermitian operators.[619] This excludes the possibility of invoking dynamically relevant properties that are representable in terms of real-valued continuous functions, as in the Bohmian case. If such a possibility is allowed, then it turns out that the Indistinguishability Postulate can still be satisfied, while the particles possess distinct, dynamically relevant properties (their positions) which allow them to be distinguished and hence individuated.[620]PII is thus preserved in more or less the same way it is in the classical situation—via the spatio-temporal positions of the particles.

However, the ontological picture is not quite as straightforward as it might seem. Certain interference experiments—interpreted within this framework—seem to imply that the properties of the particles actually belong to the pilot wave rather than the particles themselves, thus undermining the classical nature of the ontology.[621] If such properties are rendered as 'non-local' as the state-dependent ones, it is difficult to see how they can be regarded as possessed by individual particles. Here a 'principle of generosity' has been proposed which assigns these properties to *both* the particle and the pilot wave[622] and which retains some element of classicality. Again we see how the attempt to retain a sense of individuality in this context has a cost: intrinsic properties are no longer solely 'possessed' by the particles themselves and hence the relevant sense of 'object' has been broadened in this case.[623]

Further criticisms of the view that the Bohm interpretation is philosophically classical can be found in Bedard (1999). It is interesting, from the perspective of our discussion, that one of Bedard's criticisms is that the emphasis on particles and their spatio-temporal trajectories fails to account for atomic and molecular bonding, where the nature and effects of the wave field cannot be ignored. Of course, the quantum mechanical explanation of chemical bonding crucially hinges on the exchange integral which effectively embodies the Indistinguishability Postulate (it is this integral that the wave field encodes). And when it comes to understanding the latter, the advocate

[619] op. cit. Ch. 5, p. 76.

[620] Belousek shows how the Indistinguishability Postulate can be formulated within this approach; see Belousek preprint.

[621] Brown 1994.

[622] Brown *et al.*, op. cit., pp. 7–8.

[623] Brown *et al.* suggest the relevant sense of individuality should be grounded in these intrinsic properties (ibid., p. 7), but this is obviously problematic, not just because of the above concerns but also because Brown *et al.* are looking in the wrong place for individuality: particles of the same kind will possess the same intrinsic properties and are indistinguishable in that sense. It seems more appropriate in this context to insist that the particles are both distinguished and individuated by their distinct spatio-temporal trajectories, as in the classical case (there is still the problem of bosons which start out on coincident trajectories).

of the Bohm view can simply appeal to the accessibility constraints discussed previously. Hence, although the trajectories themselves might not play a role in explaining bonding, the metaphysics of individuality certainly can.[624]

Nevertheless, one might wonder if some attenuated form of Space-Time Individuality might be retained within the orthodox interpretation of quantum mechanics. We recall the way in which this form of individuality is manifested within classical statistical mechanics via the phase space formulation. And, indeed, there does exist a quantum mechanical version of the phase space approach to particle statistics, although it tends not to feature in the philosophical discussions of particle individuality.

Again, it is worth recalling some history here. Schrödinger's early attempt to give a broadly classical interpretation of the new quantum mechanics foundered on the point that the appropriate space—known as configuration space—for a many-particle wave function had to be multi-dimensional. Even then, as Einstein pointed out in 1927, use of the full configuration space formed by the *N*-fold Cartesian product of three-dimensional Euclidean space appeared to conflict with the new quantum statistics insofar as within this full space, configurations related by a particle permutation are regarded as distinct. The standard technique, of course, is to identify points corresponding to such a permutation and thereby construct what is known as the 'reduced quotient space' formed by the action of the permutation group on the full configuration space. Appropriate quantum conditions can then be applied.[625] In the context of this kind of approach,

> … the mutual indistinguishability [in the physicists' sense] of the particles is coded into the topology of configuration space itself, and the different statistical types then correspond to different choices of boundary conditions on the wave function ….[626]

This obviously fits nicely with the general understanding of the Indistinguishability Postulate we have been exploring in this chapter. However, two issues immediately come up: first of all, and most importantly, to what extent do we recover a quantum version of Space-Time Individuality within this configuration approach; secondly, does it still give us the full panoply of different kinds of statistics?

That these issues are related can be seen if we think about the problem of collisions: the reduced configuration space is not in general a smooth manifold since it possesses singular points where two or more particles coincide. This leads to two technical difficulties: first, it is not clear how one might define the

[624] For a critical response to Bedard, see Dickson 2000.

[625] See Leinaas and Myrheim 1977.

[626] Bourdeau and Sorkin 1992, p. 687.

relevant Hamiltonian at such singularities[627] and, secondly, it would appear that their presence implies Bose-Einstein statistics only. The obvious, and now standard, solution is to simply remove from the configuration space the subcomplex consisting of all such coincidence points, yielding a smooth manifold.[628] Imbo, Shah Imbo and Sudarshan have shown that within this framework one can obtain not merely ordinary statistics, parastatistics and fractional or anyon statistics but even more exotic forms which they call 'ambistatistics' and 'fractional ambistatistics'.[629]

On the other hand, it has been suggested that one of the advantages of the configuration space approach is that it actually excludes the possibility of non-standard statistics.[630] This claim is based on the work of Leinaas and Myrheim (1977), which apparently demonstrates that for a space of dimension 3 or greater only the standard Bose-Einstein and Fermi-Dirac statistics are possible. Brown *et. al.* take this to undermine the idea of symmetrization conditions being imposed as kinds of 'initial conditions':

> … the (anti-)symmetrisation condition on the wave function is now seen to be related to the dimensionality of space, in contrast to the Messiah and Greenberg analysis wherein the (anti-)symmetrisation condition receives the status of a postulate.[631]

If this were correct, then it would obviously undermine the kind of approach we have adopted here. However, the claim is problematic. It involves two crucial steps: the first takes into account the topology of the reduced configuration space and the basic point is that in the three-dimensional case, the reduced space (minus the singularity) is doubly connected, whereas in the case of two-dimensions it is infinitely connected. The next all-important step is to apply an appropriate quantization procedure and here Leinaas and Myrheim follow what they take to be the simplest way, namely the 'Schrödinger quantization scheme'.[632] What this means—no surprises here—is that for each point in the configuration space a one-dimensional complex Hilbert space is introduced and the state of the system is then described by a continuum of vectors

[627] Bourdeau and Sorkin 1992, p. 687.

[628] The relevant group for $n$ particles is then the $n$-string braid group and the irreducible unitary representations of this group can be used to label the different statistics that are possible.

[629] Imbo, Shah Imbo and Sudarshan 1990. Recalling our earlier remarks regarding the existence of particles obeying non-standard statistics, it is interesting that at the conclusion of their paper, Imbo, Shah Imbo and Sudarshan note a possible statistics problem for the putative particles associated with a gravitational field—geons—and suggest that it can be solved through the application of fractional ambistatistics (op. cit., p. 107). There is a nice, if obvious, comparison here with Greenberg's application of parastatistics to the problem of quark statistics.

[630] Brown *et al.*1999.

[631] Brown *et al.*, op. cit., p. 3.

[632] op. cit., p. 10.

spanning this Hilbert space; that is, "…Ψ is assumed to be a single-valued function over the configuration space, whose function value Ψ(*x*) at the point *x* is a vector in [the Hilbert space]".[633] If Ψ undergoes parallel transport around the singularity it transforms into *P*Ψ, where *P* is a (familiar) linear, unitary operator acting in the Hilbert space. And—here's the crucial point—since the latter is one-dimensional, *P* is just the well-known phase factor *expt[iy]*, where *y* is real and independent of the point *x*. For bosons and fermions $y = o$ and π, respectively, but in the cases of one-dimensional and two-dimensional configuration spaces a continuum of intermediate statistics is permitted. In the three-dimensional case, however, the fact that the space is doubly connected means that the further condition must be imposed on *P* that $P^2 = 1$ so that $P = \pm 1$ and, of course, only Bose-Einstein and Fermi-Dirac statistics are allowed. In other words, non-standard statistics are eliminated not just because the relevant configuration space is doubly connected but because a standard quantization procedure incorporating one-dimensional Hilbert spaces is assumed. From the group-theoretical perspective this amounts to allowing one-dimensional representations only and so it should come as no surprise that paraand ambi-statistics cannot arise. Effectively what Leinaas and Myrheim have done is to ignore the 'kinematical ambiguity' inherent in the quantization procedure which derives from the (mathematical) fact that the set of irreducible representations of the permutation group contains not just the trivial representation manifested above, but also others corresponding to exotic statistics.[634]

What about the more philosophical issue concerning particle individuality? Let us consider, first of all, the justification for the removal of the coincidence points. An obvious rationale for this would be to appeal to the Impenetrability Assumption. This in turn can be understood as due to certain repulsive forces holding between the particles. Such a conjecture has been made in the case of anyons[635] and more generally this has been taken to confer a further advantage on the configuration space approach, in the sense that

> … it allows particle statistics to be understood as a kind of 'force' in essence similar to other interactions with a topological character, like the interaction between an electric and magnetic charge in three spatial dimensions, or the type of interaction in two dimensions which is responsible for the Bohm-Aharonov effect and fractional statistics.[636]

[633] Ibid., p. 13.

[634] Imbo *et al.*, pp. 103–4.

[635] See Aitchison and Mavromatos 1991.

[636] Bourdeau and Sorkin op. cit., p. 687.

Again, we can't help but recall the relevant history, explored in Chapter 3. The suggestion that the non-classical aspects of quantum statistics reflects a lack of statistical independence and hence a kind of correlation between the particles goes back to Ehrenfest's early reflections on Planck's work and crops up again and again in the literature. And as we noted above, Reichenbach argued that such correlations—taken realistically in this sense—represent causal anomalies in the behaviour of the particles: for bosons these anomalies consist in a mutual dependence in the motions of the particles which could be characterized as a form of action-at-a-distance; for fermions, the anomaly is expressed in the Exclusion Principle if this is interpreted in terms of an interparticle force.[637] As far as Reichenbach himself was concerned, such acausal interactions should be rejected, which meant adopting the Received View.

Moving on from this issue of acausal anomalies, Brown *et al.* reject this understanding of impenetrability as suspiciously '*ad hoc*' in the general case of indistinguishable particles.[638] Shifting to the framework of the Bohm interpretation removes this ad hocness. Since the guidance equations are first-order, the trajectories of two particles which are non-coincident to begin with will never coincide. In effect the impenetrability of the particles is built into these equations and the singularity points remain inaccessible. Thus the topological approach and de Broglie-Bohm interpretation fit nicely together:

> … within the topological approach to identical particles the removal of the set … of coincidence points from the reduced configuration space … thus follows naturally from de Broglie-Bohm dynamics as it is defined in the full space ….[639]

Our point is not to advocate the Bohmian approach but merely to emphasize that the configuration space approach can also provide a suitable formal framework for this package of individual particles subject to accessibility constraints (we recall Bourdeau and Sorkin's remark above about different choices of boundary condition on Ψ). However, it is also important to recognize that this topological approach offers nothing new, philosophically speaking, since one could also maintain the alternative view that the particles are non-individuals. Indeed, one of the motivations given by Leinaas and Myrheim is that within this approach one can dispense with the whole business of

[637] Reichenbach op. cit., pp. 234–5. For further discussion see French 1985, pp. 193–5.

[638] op. cit., p. 5.

[639] Ibid., p. 6.

introducing particle labels and then effectively emasculating their ontological force by imposing appropriate symmetry constraints.[640] Of course, there is still the issue of the singular points, whose removal on the grounds of impenetrability considerations meshes so nicely with the metaphysics of individuality. One possibility is to tackle the problem of collisions directly. Bourdeau and Sorkin, for example,[641] focus on the Hamiltonian, which they require to be self-adjoint and in the two-dimensional case they show that for fermions, the self-adjoint extension of the Hamiltonian to cover the singularities is unique, so that collisions are strictly forbidden, whereas in the case of both Bose-Einstein and fractional statistics there are a range of alternative extensions, some of which allow collisions but some of which do not. By requiring that the wave function remains finite at the coincident point, they argue that a unique choice of Hamiltonian can then be made and it turns out that collisions are allowed only in the case of Bose-Einstein statistics. Thus, whereas for fermions it doesn't really matter whether the singular points are retained or not, for bosons and anyons, on this account, it does, since these points are either the locations of collisions in the boson case or the locations of vanishing $\Psi$ for anyons. There is clearly more to say about these and related issues—such as the generalization of this line to higher dimensions and parastatistics, for example[642]—but our aim is just to indicate how the configuration space approach can be made consistent with both metaphysical packages, albeit at a certain cost.

There is a further cost associated with the account of quantum particles as individuals and this is revealed when we consider the implications of quantum entanglement. We shall begin by outlining the argument that such entanglement implies a form of non-individuality, before indicating how this is not, in fact, decisive.

## 4.4 INDIVIDUALITY, BELL AND NON-SUPERVENIENT RELATIONS

The de Broglie-Bohm position is a form of 'hidden variables' interpretation. Such interpretations typically claim that quantum mechanical systems can be characterized by some set of 'hidden' factors (position in the de Broglie-Bohm

[640] op. cit., p. 2; cf. also Bourdeau and Sorkin op. cit., p. 687.

[641] op. cit.

[642] Ibid., p. 694.

case) which eliminate the ontological indeterminacy represented by the wave function. They were originally thought to be ruled out by von Neumann's anti-hidden variables proofs but these contain certain loopholes, which were subsequently exploited by Bohm for example. In the 1960s the cost of adopting such an interpretation was revealed by Bell via his eponymous theorem: if we assume some form of 'locality' condition—broadly understood as the constraint that the possessed value of some attribute belonging to one member of a pair, say, of previously interacting but now spatially separated systems, should not be affected by an alteration in the physical arrangement of an apparatus interacting with the other member—then in the context of a hidden variables interpretation one can derive an inequality which is violated by quantum mechanics.[643] Perhaps the most important virtue of this fundamentally important result is that it is experimentally testable and to date the experimental results have been in favour of quantum mechanics.[644] The implication of this result is that in such situations quantum mechanics manifests 'non-local' interactions in some sense. These situations are precisely those in which the relevant systems are in superposed or 'entangled' states and the concern is that this non-locality might bring quantum mechanics into conflict with relativistic considerations.

As originally developed, this result applied to *deterministic* hidden variables interpretations, in which the set of such variables determines the values of the outcomes of certain measurements. In response, attention shifted to indeterministic or stochastic forms of such interpretations, in which the hidden variables determine only the *probability* of an outcome. In this context it proved more difficult to generate a Bell-type inequality and appeal has to be made to two logically independent conditions:[645]

(1) 'locality': the probability for the outcome of a measurement in one wing of the apparatus is statistically independent of the *state* of the measuring device in the other wing;
(2) 'completeness': the probability for the outcome of a measurement in one wing of the apparatus is statistically independent of the *outcome* of the measurement in the other wing.

[643] See Redhead 1983.

[644] There is an enormous literature concerning Bell's Theorem and its consequences; for a review see Cushing and McMullin 1989.

[645] Jarrett 1984.

Since (1) is equivalent to the locality constraint underpinned by Special Relativity, which most (but by no means all) physicists would be reluctant to abandon, the violation of the inequality based on these conditions has led to attention being focused on (2).[646] Thus it has been noted that if it is the latter that is targeted, rather than (1), then there is the possibility of 'peaceful coexistence' with the constraints of Special Relativity.[647] However, there is still the issue of the metaphysical basis of (2) and in particular the question arises as to how we should understand its violation.

Further light has been shed on these issues by Howard's interpretation of (2) as a 'Principle of Separability' and it is this which ties these considerations to our account of quantum individuality.[648] The Principle embodies the idea that spatially separated systems possess distinct states. Its origins lie in the field-theoretic approach, where, as we saw in Chapter 2, field quantities are defined at space-time points and the (mathematical) separability of the points is inherited (ontologically) by these field quantities which in turn represent properties of systems. According to Howard, it was this notion of separability that was the real focus of Einstein's concerns in the EPR debate.[649] It is interesting, given our discussion in Chapter 3 of Born's view of objectivity in physics, that Einstein presents this notion in a 1948 letter commenting on Born's work. He writes that,

> … if one abandons the assumption that what exists in different parts of space has its own, independent, real existence, then I simply cannot see what it is that physics is meant to describe. For what is thought to be a 'system' is, after all, just a convention, and I cannot see how one could divide the world objectively in such a way that one could make statements about parts of it.[650]

Later that same year Einstein sent Born a draft of his paper in which the incompleteness of quantum mechanics, as standardly formulated, was derived from its failure to satisfy separability.[651] Again, this notion is articulated in terms of the 'independent existence' of spatially separated systems and Einstein insists that,

> [u]nless one makes this kind of assumption about the independence of the existence (the 'being-thus') of objects which are far apart from one another in space—which stems in the first place from everyday thinking—physical thinking in the familiar sense

[646] See Jarrett ibid., p. 585; Shimony 1984a and 1985.

[647] For a very clear exposition of these results, see Cushing 1989.

[648] Howard 1989.

[649] Howard 1985.

[650] Einstein to Born in Born 1971, pp. 164–5.

[651] Einstein 1948.

> would not be possible. It is also hard to see any way of formulating and testing the laws of physics unless one makes a clear distinction of this kind. This principle has been carried to extremes in the field theory by localizing the elementary objects on which it is based and which exist independently of each other, as well as the elementary laws which have been postulated for it, in the infinitely small (four-dimensional) elements of space.[652]

Born's response is worth noting.[653] He argues that Einstein's 'axiom' fails for light, which exhibits the kind of coherence which violates separability. Hence if quantum mechanics is deemed incomplete on the basis of such a violation, then the theory of light must be regarded as incomplete as well. Of course, the analogy is not entirely satisfactory: in order to obtain coherent behaviour from a light beam, we typically need to split it but in the case of particles we *assume* we have two to begin with. It is this assumption that generates the problem, as how can two entities, each with their own 'being-thus' (individuality) as it were, exhibit such coherence? This is the root of Einstein's concern but Born insists that it must simply be accepted,[654] without, however, spelling out the metaphysical implications.

These are taken up by Howard, who urges that separability should be understood as a principle of *individuality*:[655]

> … we can now understand the larger point Einstein intended to make about field theories. It is that by modeling a physical ontology upon the ontology of the mathematical manifold, we take over as a criterion for the individuation of *physical* systems and states within field theories the mathematician's criterion for the individuation of *mathematical* points. This criterion is the existence between two points of a non-vanishing interval, which gets interpreted as a three-dimensional spatial interval in classical electrodynamics, and as a four-dimensional spatio-temporal or metrical interval in general relativity. In this way, field theories—as understood by Einstein—necessarily satisfy the separability principle.[656]

Thus Howard takes separability as a necessary condition for individuality and suggests that what the violation of Bell-type inequalities shows is that, first, we must give up separability for quantum particles in entangled states

[652] Einstein 1948, pp. 170–1.

[653] Ibid., pp. 174–6.

[654] Ibid., p. 174.

[655] Howard claims that this idea of separability as a principle of individuation can be traced to Schopenhauer, situating the latter in Einstein's pantheon of personal heroes. However, despite Howard's acknowledgement of Suarez, one can detect throughout his work a conflation of distinguishability and individuality: thus, Locke, for example, is fingered as emphasizing the role of space as the 'ground of individuation', although it is clear from the passage cited that this role is that of a principle of distinguishability, whereas the 'principium individuationis' is 'existence itself' (Howard op. cit., p. 40).

[656] Howard 1989, p. 236.

and hence, secondly, that when in such states these particles cannot be regarded as individuals. In other words, Bell's Theorem implies non-individuality.[657]

How should we respond to this result? One way is to simply follow the determined Bohmian, who accepts a form of non-locality via the operation of the quantum potential and thus has no fear of Bell's Theorem. Hence we could maintain the Bohmian's sense of individual particles, grounded ultimately on Space-Time Individuality, at a cost of admitting non-locality into our metaphysical scheme. There is an alternative, however, which maintains the individuality of the particles and shifts the metaphysical peculiarities of quantum mechanics onto the relations manifested in these 'entangled' states.

The heart of this alternative lies in Teller's claim that this notion of 'entanglement' can be thought of as an expression of 'non-supervenient' relations holding between the particles.[658] Now, the notion of 'supervenience' that is used here is more typically found in discussions of the philosophies of mind and morality.[659] According to this characterization, a dyadic relation R is supervenient upon a determinable non-relational attribute P if and only if:

(a) $\Box (\forall x, y) \neg \Diamond \{R(x, y)$ and there are no determinate attributes $P_i$ and $P_j$ of determinable kind $P$ such that $P_i(x)$ and $P_j(y)\}$

(b) $\Box (\forall x, y)\{R(x, y) \supset$ there are determinate attributes $P_i$ and $P_j$ of determinate kind $P$ such that $P_i(x)$ and $P_j(y)$ and $\Box (\forall x, y)[(P_i(x)$ and $P_j(y)) \supset R(x, y)]\}$

where a physical attribute $P$ is understood as 'non-relational' if and only if it is possible that there exists an $x$ such that $x$ has $P$ and no individual physical thing wholly other than $x$ exists (in other words, non-relational attributes are independent in the sense that they can be instanced in a universe containing only

[657] The most obvious understanding of separability would be as a 'physicalized' form of the Impenetrability Assumption and hence its violation amounts to another violation of Space-Time Individuality. However, Howard himself interprets it as a property-based principle of individuality, since the quantum states characterize intrinsic properties of the particles and it is these properties which can be regarded as 'carrying' the individuality 'lost' by particles in entangled states. This is a rather strange position to adopt and has the unfortunate consequence, noted by Howard himself, that the conferral of individuality becomes dependent on the strength of the interactions between the particles. A more coherent account can be given if one were to say that this 'lost' individuality is not taken up anywhere else but that distinguishability, at least, is retained by the states, regarded as embedded in space-time.

[658] Teller 1986; for further details in the context of Bell's Theorem, see also Teller 1989.

[659] Teller 1984; an earlier attempt to introduce it into a physical context can be found in Cleland 1984 and we shall find her analysis useful in what follows. Cleland actually deploys this notion in consideration of the nature of space-time, arguing that spatio-temporal relations should be regarded as non-supervenient.

one individual). What these conditions mean can be spelled out as follows. If *R* is genuinely supervenient upon *P* then (a) implies that the relation *R* cannot possibly appear in the absence of each of its relata instancing the requisite reductive property *P*, whereas (b) states that there must exist one or more pairs of determinate monadic properties (of kind *P*) whose exemplification alone is sufficient to guarantee the appearance of *R*.

This then yields two forms of 'non-supervenience', according to whether both (a) and (b) or (b) only are violated, respectively:

> *Strong Non-Supervenience*: a relation is 'strongly' non-supervenient upon a determinable non-relational attribute if the appearance of this relation is neither dependent upon nor determined by its relata bearing the non-relational attribute concerned.
>
> *Weak Non-Supervenience*: a relation is 'weakly' non-supervenient upon a determinable non-relational attribute if the appearance of this relation is dependent upon the instantiation of the non-relational attribute in the sense that the relation could not possibly be exemplified in the absence of each of its relata separately bearing the determinable non-relational attribute in question, but there exist no determinate non-relational attributes whose manifestation is sufficient for the appearance of the relation.

The question now, of course, is whether the inter-particle relations expressed by the entangled states of quantum mechanics can be described as 'non-supervenient' in either of these senses.[660] To answer this let us recall the case of two particles distributed over two one-particle states and the state-functions given as (5), (6), (7) and (8) above. Assuming the standard Eigenstate-Eigenvalue link again, these can all be taken to represent properties of the particles. (5) and (6), although relational, can be factorized into properties of each particle separately. Thus there exist, in this case, pairs of monadic properties whose exemplification alone is sufficient to guarantee the appearance of the relations expressed by (5) and (6) and these relations satisfy condition (b) and hence also (a) of Cleland's characterization. In other words, they are not 'inherent' relations, in Teller's sense of not supervening on the relevant non-relational properties.

However, it is not so immediately clear that this is also true for the kinds of relations expressed by (7) and (8). We have already seen that one cannot ascribe

[660] See French 1989c. Howard writes, in his analysis of Bell-type situations, that "… we tacitly assume … that the joint state of any set of such point-systems is wholly determined by the states of its constituents" (1989, p. 235), which amounts to an assertion of supervenience.

to each particle an individual state function on the basis of either (7) or (8), since these state functions are not the product of the separate state functions of the particles. It is this, of course, which reveals the peculiar non-classical holism of quantum mechanics. We recall that the odd nature of these states is revealed when one considers correlation measurements between observables relating to each particle, such as joint measurements in the two-particle system relating to the probability of finding both particles in the same one-particle state, as we discussed in the section on PII. These statistical correlations—to the effect that, for example, if one fermion is occupying a given state, all other fermions are excluded from that state—express a relation between, or relational property of, the particles, in Cleland's sense. The question now is, are such relations 'inherent' or not?

A negative answer could be given if one were willing to entertain non-classical forces coupling between certain properties of the particles, as exemplified by the so-called 'exchange forces' which we've already mentioned. This would then effectively reduce the relational properties represented by (7) and (8) to non-relational attributes of the relata.[661] We have already indicated the objections that have been made to such approaches. First of all, such forces would have to be non-local, acting 'at a distance', and possibly would have to be acausal as well.[662] This may not unduly perturb the advocates of this sort of line, as they are generally happy to accept non-local, but not necessarily acausal interactions.[663] Secondly, this sort of approach is typically accompanied by a further—and sometimes quite dramatic—expansion of our ontology; thus, in addition to the Bohm potential we have these peculiar forces, not manifested in terms of the familiar four (or three) and mediated by some kind of 'sub-aether'.[664] At this point, the price of reducing our relations of entanglement may seem unacceptably high.[665]

[661] See, for example, Cufaro-Petroni *et al.*1984.

[662] Reichenbach op. cit.

[663] Cufaro-Petroni op. cit.

[664] Ibid.

[665] Nonetheless, these attempts undercut Teller's claim that the introduction of hidden variables cannot reduce the relations exemplified by (7) and (8) to non-relational properties of the particles, since they violate the results of Bell and Kochen and Specker (Teller 1986, p. 79). If violation of Bell's result is understood to imply some form of non-locality, then this is happily swallowed by supporters of these approaches. As for the famous Kochen and Specker result, this can be circumvented by introducing so-called 'contextual' hidden variables, where the value of the hidden variable is given with respect to a certain local environmental 'context' (see Shimony 1984b). Teller argues that this amounts to admitting non-reducible relational properties but we have to be a little careful here. A contextual hidden variable is 'relational' in the sense that it holds between the particle and the relevant context. But this is not the same thing as being relational in the sense of holding, whether superveniently or not, between two or more particles. The difference is as regards the relata involved and this is an important difference; see French 1989c and Belousek 2003.

If we are unwilling to pay this price, then a straightforward argument can be given to the effect that these quantum statistical correlations manifest relations which are 'inherent' in the sense of not supervening upon properties of the relata. The argument runs as follows:[666]

(1) state functions representing such correlations cannot be reduced to simple products of the functions for each particle separately;
(2) states represented by (7) or (8) and those represented by (5) and (6) are distinct so that a system 'in' one is never 'in' the other;
(3) a system possesses a certain property if and only if it is in the corresponding eigenstate; therefore,
(4) a system in the eigenstate represented by a superposition or entanglement possesses a relational property which cannot be reduced to a monadic property of the particles.

In other words, the relations manifested by these statistical correlations are irreducible to properties represented by, for example, (5) and (6) and do not supervene upon the latter. Let us now return to our question, are they 'strongly' non-supervenient or only 'weakly' so?

It is worth comparing the quantum case with Cleland's own example of spatio-temporal relations. She argues that these are only weakly non-supervenient upon non-relational attributes of material bodies, since such relations are *dependent* on non-relational attributes such as size and shape, thus satisfying (a) above, but are not *determined* by such attributes, thus violating (b). However, this conclusion crucially hinges on our being able to imagine a situation, a possible world, in which the size and/or shape of an object are stripped away and in such a case we would say that the object ceases to be at any distance from another. The important point to note is that it is because an object can possess both size/shape and certain inter-object relational properties simultaneously that we have to move to a possible world in which the former are removed in order to see if the latter disappear as well. In quantum mechanics the situation is quite different, as our arguments above show. In this case the properties represented by (6) and (7), for example, are mutually exclusive in the sense that a system in one cannot simultaneously be in the other. Hence there is no need to invoke possible worlds here since it is already a 'built in' feature of the theory that these properties are independent

[666] Teller 1986, pp. 78–9.

of one another. This, of course, is the thrust of the argument that (7) cannot be reduced to (5) or (6). As Teller remarks, referring to the Bell-type situation:

> … the superposition characterises an independently identifiable property with distinctive experimental implications for the 1-2 system as a whole.[667]

If the properties represented by (7) and (8) are not dependent upon those represented by (5) or (6), then clearly they are not determined by them either. Therefore, in contrast with Cleland's inter-object distance relations, the quantum mechanical entangled or correlation relations are *strongly* non-supervenient with respect to the relevant non-relational properties. Furthermore, if the bundle view of individuality is rejected, then these relations are strongly non-supervenient with respect to the intrinsic properties of the particles as well. The conclusion, then, is that the 'entangled' relations of quantum mechanics are strongly non-supervenient with respect to non-relational or monadic properties in general.

It is hard to avoid the implication that such relations must be regarded as having a physical reality *qua* relations, transcending all the other properties of the relata, and should be taken as basic constituents of the world along with particulars and monadic properties.[668] Teller himself calls this view 'Relational Holism' and contrasts it with what he calls 'Particularism', which states that:

> … the world is composed of individuals, that the individuals have nonrelational properties and that all relations between individuals supervene on the nonrelational properties of the relata.[669]

What quantum mechanics tells us, he insists, is that particularism is simply wrong and that we must endorse 'Relational Holism' and accept that there are non-supervenient relations. Nevertheless, the metaphysical implications have yet to be fully spelt out. Perhaps the most obvious way of embarking

[667] op. cit., p. 80. In the EPR case the properties originally chosen were position and momentum, so that (7) or (8) represent a superposition of positions for particles 1 and 2 which constitutes an eigenstate of the distance between them and is also an eigenstate of their total momentum, but not of any specific positions or momenta of 1 or 2 separately. Again the superposition represents a property which is independent of the non-relational properties of the particles. In this case if we were to accept Cleland's conclusion that spatial positions are given in terms of (weakly) non-supervenient spatial relations we would have an instance of the circularity previously noted.

[668] Cleland tries to make the same point for her weakly non-supervenient spatial relations (op. cit., pp. 36–7) but since these are dependent upon certain non-relational attributes, it is difficult to see how they could be regarded as independently existing basic constituents in the same sense. Of course, this does not preclude the possibility of an alternative analysis of spatio-temporal relations arriving at a similar conclusion to ours.

[669] Teller 1989, p. 213. Teller explicitly takes 'particularism' as an expression of Einstein's understanding of separability (cf. p. 215, fn. 8).

on such a project would be to start with the traditional account of relations in general.

As is well known, this issue of how we are to understand relations *ontologically* is one that crops up again and again throughout the history of philosophy. The commonly held view that relations should be regarded as reducible to non-relational properties is reflected in the claim that every statement of the form $R(x, y)$ can be reduced to one of the form $P(x)$, where $P$ stands for 'being $R$ to $y$'; that is, relational statements can always be 'reduced' to statements of subject-predicate form. Of course, as Cleland notes,[670] there is something rather unsatisfactory about this way of simply 'parsing out' relations. For one thing, it amounts to nothing more than a restructuring of a statement so that it contains a predicate which *looks* as if it refers to a non-relational property but contains an ineliminable reference to a particular, unlike genuine non-relational properties which are exemplified by particulars, rather than having them as constituents.

One response would be to simply deny the connection between physical and logical reduction in this case—a response which acquires a certain plausibility in the context of the quantum logic programme.[671] Teller himself drew on Stairs's realist quantum logic approach, which attempts to formalize the idea that,

> … there can be facts about a pair of quantum systems which are, in a clearly specifiable sense, about the whole system but are neither reducible to nor implied by facts about the parts.[672]

Quantum correlations are then explained by appealing to the 'unusual' relations which quantum systems bear to one another and the way these then relate to probability.[673] One can interpret this programme as an attempt to bring the logical situation back in line with the physical one.

Are there any other non-supervenient relations? The fact is, spatio-temporal relations aside, the relations which are of interest to physicists all supervene on the intrinsic properties of the particles. From the phenomenological perspective, Weyl suggested that this may have its origin "… within the domain of sense data, which—it is true—can yield but quality and not relation".[674] This might explain why relations between 'classical' objects are not regarded as irreducible in Teller's sense, but it should not be taken as binding in the quantum realm.

[670] op. cit., p. 20.

[671] See Dickson 2001.

[672] Stairs 1984, p. 357.

[673] Ibid., p. 358.

[674] Weyl 1963, p. 4.

James, on the other hand, grounded his 'radical empiricism' on the 'statement of fact' that "… the relations between things, conjunctive as well as disjunctive, are just as much matters of direct particular experience, neither more nor less so, than the things themselves".[675] However, calling a composite such as a quantum system in an entangled state an irreducible 'concatenated union',[676] sheds little further light on the nature of this union.

More recently, Ladyman has suggested that this view of entangled relations might be naturally understood in a structuralist context.[677] However, this seems clearly different from the case of Saunders's approach above, in that here the individuality of the particles is not understood as constituted by the relations but as given independently of them. Nevertheless, by broadening our metaphysical pantheon, the idea of non-supervenient relations might be seen as opening up ontological space for structuralist conceptions.[678] Such conceptions have also drawn support from the contrast between metaphysical 'packages' we have been exploring here. In our final section we shall look at the nature of that contrast a little more closely.

## 4.5 THE UNDERDETERMINATION OF METAPHYSICS BY PHYSICS

We have now reviewed a number of different ways in which one might 'flesh out' the metaphysical package of particles-as-individuals, together with some of the more obvious problems that this package must face. If it is agreed that these problems can be overcome, then the conclusion is that the formalism can be taken to support two very different metaphysical positions, one in which the particles are regarded as 'non-individuals' in some sense and another in which they are regarded as (philosophically) classical individuals for which certain sets of states are rendered inaccessible.

There is, then, a kind of underdetermination of the metaphysics by the physics and thus good reason to be sceptical of claims that physics forces us to

[675] James 1932, p. xii.

[676] James 1909, p. 358.

[677] Ladyman 1998.

[678] Humphreys has situated quantum non-supervenient relations within his framework for representing emergent properties (Humphreys 1997, pp. 15–16). Not only does this give a useful example for his purposes but it provides a further metaphysical context in terms of which quantum entanglement can be elaborated.

drop the view of elementary particles as named individuals.[679] It then follows that the claims of certain 'naturalist' philosophers that we can simply read our metaphysics off the relevant physics are highly suspect. Quine, for example, has drawn on the Received View to argue that developments in physics this century support the evaporation of physical objects—at foundation, elementary particles—into nothing more than regions of space-time bearing certain properties.[680] Similarly, and more recently, Resnick has sought support for his 'structuralist' view of mathematics in the same developments (1990). In particular he claims that epistemic differences between mathematical and physical objects are blurred by recent physics.[681] Even Lucas can be set among these odd bedfellows as he argues that the disappearance of transcendental individuality from the 'categories of ultimate reality'[682] has contributed to the opening up of 'new vistas of rationality' and supports, in particular, an anti-reductionist account of theory autonomy.[683]

More generally Fine, in his 1988 Presidential Address to the Philosophy of Science Association, urged the assembled cohorts to "… actively engage philosophy with on-going science" and reminded us of "… the potential in science itself for addressing virtually all the sorts of interpretative questions and issues that philosophy traditionally pursues".[684] Someone who has taken this latter naturalistic claim very seriously is Shapere, who has argued that science can in fact resolve traditional philosophical questions concerning, for example, identity and existence.[685] At the same 1988 meeting, he suggested that "… by being internalized into the scientific process, even such concepts as explanation and existence can be subject to alteration in the light of what we learn".[686] The problem is, it is not always clear what it is that physics teaches us!

[679] French and Redhead 1988; French 1989a; similar conclusions have been reached by van Fraassen 1991 and Huggett 1997.

[680] Quine 1976; see also Dalla Chiara and Toraldo di Francia 1995.

[681] Hale, in an accompanying piece, has pushed this line even further, drawing on the discovery of anti-matter and Heisenberg's 'anti-foundationalist' view of elementary particles (1990). However, her invocation of the history of modern physics on this point fails to take account of the decline of *S*-matrix theory and the rise of the gauged field theoretic formulation.

[682] Lucas 1993, p. 2.

[683] "Explanations are inherently universalisable, and if the physical universe is one of qualitatively identical features that cannot, even in principle, be numerically distinguished, then the explanations offered by other disciplines are ones that cannot, even in principle, be improved upon by a fuller physical explanation. Indistinguishability and indeterminism imply a looseness of fit on the part of physical explanation which take away its Procrustean character. The new world-view makes room for there being different sciences which are autonomous without invoking any mysterious causal powers beyond the reach of physical investigation" (Lucas, op. cit., p. 8).

[684] Fine 1989.

[685] Shapere 1990.

[686] Shapere 1989.

Of course, in some cases the lesson does seem pretty straightforward, even blunt: as we have indicated, a strong case can be made that the Principle of Identity of Indiscernibles can be ruled out in the quantum domain. Such conclusions give the lie to the claim that physics cannot yield *any* metaphysical conclusions at all. And indeed, the fact that the early defenders of the Received View, together with philosophers of the stature of Quine, for example, have attempted to draw such conclusions suggests that the establishment of this metaphysical underdetermination is a significant result. Those readers who are familiar with the realism–antirealism debate in the philosophy of science will, of course, recognize this kind of conclusion. In that context it is theories themselves which are underdetermined by the empirical data and this has been deployed by the anti-realist in order to undermine realist claims that we can believe a particular theory to be true, on the basis of its explanation of such data. (As we will see shortly, this metaphysical underdetermination has also been taken to have implications for realism.) The history of the underdetermination of theories by data can be traced back to the debates of the early twentieth century concerning the nature of space-time. In these debates a 'conventionalist' view was often adopted, whereby the particular view of space-time one held was understood to be based on an initial convention concerning measurement. Similarly, our metaphysical underdetermination in the quantum context has also been linked to a form of conventionalism. Belousek, for example, has insisted that,

> Whether or not identical particles are (in)distinguishable [where by this he means (non-) individuality] … is conventional in the (Duhemian-Einsteinian) sense that is justified ultimately by an epistemologically prior choice between two (or more) observationally equivalent sets of initial hypotheses, a choice that is necessitated by neither logic nor experiment, but rather is guided by normative-methodological criteria of theory appraisal.[687]

This conclusion has been disputed on the grounds that use of the term 'conventional' in this context suggests that there is something 'arbitrary' in the metaphysical packages above, whereas they are arrived at on the basis of reasoning which is better described as 'fallible'.[688] But this is not the intention at all: again, consider the case of conventionalism in space-time physics. There the choice of an initial measurement standard was not at all taken to be determined arbitrarily or on the basis of an epistemological whim. The point

[687] Belousek 2000, p. 39.

[688] Teller and Redhead 2000, p. 956.

was, the physics itself permitted different standards—supported by different lines of fallible reasoning—to be applied and likewise in the quantum case, adopting a different understanding of the Indistinguishability Principle leads to a different metaphysical position. Describing the adoption of that different understanding as 'conventional' should in no way be taken to imply that it is arbitrary.

Having established this last point, let us move on to consider whether there are any reasons for preferring one metaphysical package over another; or, putting it in the form of a question, is there any way of breaking this underdetermination?

One option is to note that the individuals package depends on such metaphysically problematic notions as that of substance and primitive thisness. However, this will cut little ice with defenders of such concepts, who will point a similarly accusing finger at the alternative notion of a 'non-individual'. Indeed, they can argue that at least the former package allows us to deploy our standard logical and set-theoretical resources, whereas it is not clear how the idea of non-individuality is even expressible in formal terms.[689] Subsequent chapters of the present work can be taken as an attempt to respond to this argument.

A more robust line would be to claim that the non-individuals package meshes better with the framework of quantum field theory and is thus to be preferred on these grounds alone.[690] There are two aspects to this 'meshing' which need to be separated. The first emphasizes that on standard accounts of quantum field theory particle indices are not assigned from the word go and thus particle permutations are undefined.[691] However, state spaces for QFT can be constructed which do involve such indices. Thus van Fraassen has insisted that appealing to QFT cannot resolve the underdetermination, since this theory is,

> … equivalent to a somewhat enriched and elegantly stated theory of [individual] particles. That we can take it as a description of a world that is particle-less only masquerades as an incompatible alternative.[692]

[689] van Fraassen—who, as we have indicated, also defends a kind of metaphysical underdetermination—declares himself to be 'more sanguine about the applicability of classical logic and set theory' in this situation (1991, p. 430). However, as an anti-realist who holds each horn of the underdetermination to be merely one way the world could be, he can afford to be sanguine!

[690] Post op. cit.

[691] Redhead and Teller 1991 and 1992; Teller 1995.

[692] van Fraassen 1991, p. 436.

The basis for such a claim is the relationship of representation which is purported to hold between models of QFT and 'concrete' constructions of Fock space carried out within de Muynck's 'labelled particle' approach.[693] This suggestion has been criticized by Butterfield, who has cast doubt on the empirical equivalence that is supposed to exist between QFT and many-particle quantum theory.[694] More relevantly in this context, he has suggested that the existence, within QFT, of states that are superpositions of particle number render van Fraassen's claim false. The idea here is that on any reasonable account of individuality, the number of individuals must be definite. To deny the latter leaves one open to the objection that the entities concerned are not individuals at all.[695]

We shall return to these issues in Chapter 9, where we shall see that difficulties emerge in the quantum field theoretic context as well.

The second aspect of this 'meshing' between the particles-as-non-individuals package and QFT is more *methodological* in nature. If the particles are conceived of as individuals subject to restrictions on the sets of states they may occupy, then those states that are inaccessible to the particles of a particular kind correspond to just so much 'surplus structure' in our theoretical description. In particular, if the particles are regarded as individuals and particle indices introduced, then it is entirely mysterious as to why, in addition to the standard symmetrized, anti-symmetrized and mixed-symmetric (corresponding to parastatistics) combinations, a particular subset of these inaccessible, surplus states, namely those that are non-symmetric, are not only not realized in nature but appear to serve no useful theoretical purpose. They might be thought of as both empirically and theoretically 'surplus'. Applying the general methodological principle that a theory which does not contain such surplus structure is to be preferred over one that does, Redhead and Teller conclude that we have grounds for preferring the non-individuals approach and this mystery simply does not arise.[696]

Now, this methodological principle of 'avoid surplus structure' is an interesting one. The phrase itself derives from a paper by Redhead in which he used this notion to explore the idea that when a theory of physics is embedded

[693] de Muynck 1975.

[694] Butterfield 1993; de Muynck himself argued that the standard theory and his own formalism were, in fact, inequivalent.

[695] For further discussion see Huggett 1995. There is also the further argument that the Fock space representation itself fails when it comes to interacting QFT; we shall touch on this in Chapter 9.

[696] Redhead and Teller 1991 and 1992; Teller 1995.

in some mathematical framework, we often get more mathematics than we need to express that particular theory.[697] Giving a physical interpretation to certain elements of this surplus mathematical structure can then lead to fruitful extensions of the theory, as in the case of the positive and negative energy solutions of the Dirac equation, for example. Thus in its original incarnation, the notion of surplus structure was introduced as a positive aspect of the applicability of mathematics to physics, which embodied the heuristic advantages of introducing mathematical structures. Somehow this was transformed into a negative methodological principle whereby a package which is taken to contain less surplus structure than another is deemed to be more acceptable. Given the original heuristic understanding, however, the dangers of adopting such a methodological principle are obvious.

Consider the case of parastatistics, for example. If physicists had adopted Redhead and Teller's principle, then paraparticle states would have been ruled out as just so much surplus structure. Yet, as we indicated in Chapter 3, the exploration of the possible application of parastatistics to quarks was absolutely crucial for the early development of quantum chromodynamics. Teller does appear to acknowledge this point[698] but insists that, leaving the parastatistics case to one side, other, non-standard statistics have not found such a heuristic role and count as just so much surplus baggage. Even more robustly, in their reply to Belousek's 'conventionalist' understanding of the under- determination, Redhead and Teller claim that the general quantum mechanical formalism, 'together with an extraordinarily large array of experimental facts', support the conclusion that only Bose-Einstein and Fermi-Dirac statistics exist in nature.[699] The kinds of 'facts' they have in mind here are the following: if photons, for example, did not obey Bose-Einstein statistics, then the absorption and emission of light would be very different; if electrons were to exist in superpositions of symmetric and anti-symmetric states, then all the elements in the periodic table from lithium on up would possess states in which the 1s shell would have more than two electrons and hence the chemical properties of these elements would be very different.

This is a strong claim, but it can be contested. It turns out there is a large body of experimental and theoretical work exploring the possibility that 'standard' quantum statistics might be violated, albeit if only at the margins, as it were.[700] Greenberg and Messiah pointed out as early as 1964 that the

[697] Redhead 1975.

[698] See Teller 1995.

[699] Teller and Redhead 2000, p. 956.

[700] See, e.g., the recent collection by Hilborn and Tino 2000.

possibility that known particles, such as electrons, might obey parastatistics could be easily ruled out by consideration of collision experiments, for example, or simply examining the helium spectrum. Nevertheless, the possibility remains that such particles might exhibit small deviations from standard quantum statistics under certain circumstances. Exploration of this possibility has proceeded in both experimental and theoretical directions.[701] With regard to the theoretical, Greenberg and Mohapatra have proposed a form of 'paronic' statistics, according to which two indistinguishable fermions, say, can be in the normal antisymmetric state with a certain probability and in the anomalous symmetric state with the related probability.[702] Unfortunately, as they acknowledge, there are difficulties in constructing a quantum field theoretic version of such a theory, since the representation of number operators and creation and annihilation operators in a positive metric Hilbert space rules out all but standard and parastatistics. Nevertheless, the relationship between paronic theory and experimental violations of the standard statistics has been pursued at the 'phenomenological' level in the context of quantum mechanics.[703]

Greenberg himself has attempted to introduce an alternative operator formalism which accommodates small violations of standard statistics through the introduction of a parameter, $q$, which can occupy intermediate values between $-1$ and $+1$.[704] According to the formalism of these $q$-mutators, there is a probability of $(1 + q)/2$ for two particles to show bosonic behaviour and a probability of $(1 - q)/2$ that they would show fermionic behaviour. For electrons, say, small violations of Fermi-Dirac statistics would be represented by $q$ being close to $-1$.

The objective of much of this work is to test the experimental and theoretical validity of the spin-statistics theorem which reduces the statistical behaviour of an assembly of particles to one of their intrinsic properties, namely spin; thus, particles with integral spin should obey Bose-Einstein statistics and those with half-integral spin should obey Fermi-Dirac. Obviously, if the theorem could be proven then non-standard statistics would be ruled out;[705] unfortunately, no satisfactory proof has been forthcoming.[706] Hence the situation

[701] For further discussion of the experimental side, see Greenberg 1992 and Hilborn and Yuca 2002.

[702] Greenberg and Mohapatra 1987.

[703] Greenberg and Mohapatra 1989.

[704] Greenberg 1990; see also Hilborn and Yuca 2002, pp. 35–40.

[705] See Redhead and Teller 1992, p. 217, fn. 7.

[706] For a recent review of attempts, see Duck and Sudarshan 1997. In case anyone happened to be wondering, it can be shown that if the spin-statistics theorem holds, paraparticles would also satisfy it (see Greenberg and Messiah 1965).

is rather more complex than Redhead and Teller let on. It would certainly be a bad move to adopt a methodological principle that ruled out *all* non-standard wave functions, including those relevant to parastatistics, fractional statistics, ambistatistics and so on.[707] We have already noted the possible application of parastatistics to quarks and ambistatistics to geons and if violations of the spin-statistics theorem were to be discovered experimentally, then Green's $q$-mutator formalism might well be the appropriate one to adopt. If we understand surplus structure as it was originally intended, then such structure in these cases might prove rather fruitful. Of course, there may be mathematical structures—corresponding to completely non-symmetric functions for example—which are unlikely to prove at all fruitful in this way, but it is not clear whether, as a methodological principle, 'avoid surplus structure' is honed enough to cut these out whilst leaving the more interesting combinations. At the very least, we can conclude that there exists a tension between the heuristic role of surplus structure and its incorporation into methodology.

From the perspective offered by the former understanding, it should come as no surprise that when we embed quantum mechanics in group theory, we get 'surplus' representations that we don't apparently need to represent the particles that we know of.[708] One could then simply adopt the attitude that the apparent mystery is a mere fabrication: the inaccessible non-symmetric states can be ruled out as simply not physically possible.[709] The surplus structure, then, is a consequence of the representation chosen and has no further metaphysical significance.[710]

Still, the point can be pressed that this leaves a mystery as to *why* this structure is not realized and that one of the jobs of a theory is precisely to explain away such mysteries. One can draw an analogy with statistical mechanics: this theory can explain why we never observe a cold cappuccino starting to boil, whereas the quantum-particles-as-individuals view cannot explain why we never observe states of non-standard symmetry.[711] Even without taking into account the points raised above, the analogy itself is problematic. Let us reflect on our cold cup of cappuccino. Statistical mechanics does not tell us that it will never boil, but only that the probability of this occurring is extremely low. The question then reduces to that of 'why is this probability so low?' This in turn is answered by appealing to the very low number of states which correspond to the coffee boiling compared to the vast number

[707] Cf. Hilbron and Yuca 2002, pp. 39–40.

[708] French 1999.

[709] Huggett 1995.

[710] We can also find this attitude in Weyl 1931.

[711] Teller 1998.

of states for which it remains cold. If we were to press on and demand to know why there should be this disparity in the number of states accessible to the cappuccino—or, equivalently, why we should find ourselves in a region of entropy increase—then an answer might be given in terms of the initial conditions pertaining to our region of the universe. Why do we never observe entropy decreases in our cups of coffee? Because that's the way our region of the universe is, as expressed in the initial conditions. A similar line can be taken in the case of quantum statistics. Why do we never observe non-symmetric states, given this metaphysics of individual particles? Because that is the way the universe is and we should not expect quantum mechanics alone to have to explain why certain initial conditions obtain and not others. Here we recall that the symmetry of the Hamiltonian ensures that if a particle is in a state of a particular symmetry to begin with, it will remain in states of that symmetry. Hence, if non-symmetric states do not feature in the initial conditions which held at the beginning of the universe, they will remain forever inaccessible to the particles.

At some point the explanatory buck has to stop and the theory itself can't do all the work. Of course, you might prefer the alternative explanation in which non-symmetric states are ruled out from the word go, but then you might legitimately be urged to flesh out the metaphysics and logic of 'non-individuality'. No explanation of this sort comes without a price attached. In subsequent chapters we shall indicate how this cost may be off-set by setting out just such a logic. Before we do that, however, we need to address a further issue associated with the view of particles as individuals—namely the nature of the indices used to label the particles. As we shall see, regarding such labels as forms of names generates certain problems which will lead us nicely into our consideration of an appropriate formal framework.

# 5
# Names, Nomological Objects and Quasets

But above and beyond there's still one name left over
And that is the name that you never guess:
The name that no human research can discover
But the cat himself knows, and will never confess.
When you notice a cat in profound meditation
The reason, I tell you, is always the same:
His mind is engaged in a rapt contemplation
Of the thought, of the thought, of the thought of his name
His ineffable, effable, effaninefable
Deep and inscrutable singular name
Name, name, name, name, name, name.
T.S. Eliot, *The Naming of Cats*

We concluded the previous chapter by presenting a form of underdetermination between the two metaphysical packages of quantum particles as individuals and as non-individuals. In Chapters 6–8, we will elaborate a formal framework which, we claim, is capable of accommodating the latter view. In this chapter our aim is to explore certain features of the particles-as-individuals package and, in particular, the issue whether, given the way quantum mechanics (QM) treats particle labels, these particles can be regarded as *named* individuals. We recall that such labels are initially assigned but then permuted, so a particular label does not appear to 'tag' a particular particle. This has led to the suggestion that the quantum domain is the 'land of anonymity'. Before examining this suggestion, and the formal results which it motivates, we shall briefly discuss the nature and function of names.

# 5.1 THE ROLE OF NAMES IN SCIENCE

The function of names, at least, appears straightforward: it is to *refer* to an individual. When we assert that 'Max Planck was the first to quantize energy', we are using the name 'Max Planck' to refer to a specific individual. The name labels the individual. Likewise, when we write the expression $|a_1\rangle$, the label 1 is being used to refer to an individual, or so it would appear. A number of well-known philosophical issues then arise concerning reference, meaning, and the 'mechanism' by which names refer.[712]

Let's begin with an apparently straightforward view of the relationship between proper names and their bearers, according to which names simply stand for, 'tag' or 'label' objects. They have no meaning other than standing for these objects. Hence, a name does nothing more than denote its bearer and its meaning is entirely given by its reference. As Mill put it, proper names have denotation but not 'connotation', in the sense of connoting a set of specified properties. By comparison, a kind term such as 'horse' has both denotation and connotation: it denotes the kind, and connotes a set of properties which are specified in the definition of 'horse'. A proper name, on the other hand, simply and only denotes its bearer. As Wittgenstein subsequently wrote, "[a] name means an object. The object is its meaning".[713] For reasons which will shortly become clear, this can be called the 'Non-Descriptivist' view of proper names.

*How* do names refer on this view? The answer appears to be straightforward: they are like labels and are attached to or assigned to their bearers, just as labels are attached to an object; or better, just as a person's name is assigned to them at birth. There is then the further issue of how we come to *know* who or what the bearer of the name is and again, thinking, for example, of how we would come to know who 'Max Planck' refers to, we can come up with some fairly obvious answers. Already this is to fill in some of the details of the 'mechanism' of reference and we will return to these issues shortly, but first we must consider some of the well-known problems this view must face.

Let us consider again the example we mentioned in Chapter 1. 'The Morning Star is identical to the Evening Star'. This is an example of an identity statement which conveys information. How can that be, when the two names are referring to the same object? From the Non-Descriptivist perspective,

[712] See for example, Reimer 2003.

[713] Wittgenstein 1963, p. 23.

such statements are trivial. Or consider the statement 'Victor Jakob could not accept the new quantum physics',[714] which appears meaningful, but has to be regarded as meaningless on this view. And the assertion 'Victor Jakob does not exist', which appears to be not only meaningful but, intuitively, true, must be dismissed as absurd, since to be meaningful, it must presuppose the existence of that whose existence it denies! Of course, an adherent to the Non-Descriptivist position could always maintain that in each case it is our intuitions that are wrong—that indeed the *proposition* expressed by 'The Morning Star is identical to the Evening Star' is trivial but that nevertheless information is still communicated by this assertion, that although the statement 'Victor Jakob could not accept the new quantum physics' does not express a proposition, nevertheless some proposition is communicated by its assertion, and so on. However, this requires some account of how information can be communicated by trivial or meaningless propositions.

Given these difficulties, one might turn to the alternative, 'Descriptivist' view, which holds that names refer by virtue of the descriptive content associated with the name, where this content is typically specified by means of a definite description satisfied by the object answering to that name. For this 'mechanism' to work the description (or least some aspect of it) has to uniquely pick out the object being referred to. Of course, different descriptions may be associated with a name, both between individuals using the name and—more significantly perhaps—over time. Thus a historian of science may associate the description 'the person who first regarded energy as quantized' with the name 'Max Planck', while a student has the vaguer association 'some German scientist'. Obviously if there is sufficient overlap between these descriptions we would typically be happy to accept that these individuals are referring to the same person. In the case of shifts in descriptive content over time, a well-known problem arises for those who would like to be realists about physical objects: if the shift in descriptive content is sufficiently radical that there is no common element between descriptions before and after the shift—as in the case of scientific revolutions, perhaps—then it might be asked, on what grounds can it be claimed that the different descriptions are referring to the same object? Such concerns are precisely the motivation for realist-inspired developments in theories of reference, which we shall touch on shortly.

Before we do, let's just consider some of the virtues of this view. First of all, it offers an account of the *meaning* of proper names: in place of the bearer of the

[714] Professor Jakob being the fictional protagonist of McCormach's 1982.

name constituting its meaning, the Descriptivist maintains that the meaning is given by the descriptive content which fixes the reference. Thus we recall from Chapter 1 Frege's claim that the name-referent relation involves a third element—the *sense* of the name, which provides the meaning: "[a] proper name *expresses* its sense, *stands for* or *designates* its reference".[715] He argued that names essentially have a sense, but only contingently have a reference—they refer if and only if there is an object which satisfies their sense. The sense of the name can then be explicitly formulated by means of a definite description—a list of properties—and thus proper names come to be seen as equivalent in meaning to, or can be viewed as no more than, 'disguised' definite descriptions. One way of summarizing the difference between the two views we have outlined is to say that on the Non-Descriptivist theory, naming is taken to be prior to describing, whereas on the Descriptivist view, describing is prior to naming (for a name only names by describing the object it names). As Searle also noted, they point in different directions, metaphysically speaking:

> These two views are paths leading to divergent and hoary metaphysical systems. The first leads to ultimate objects of reference, the substances of the scholastics and the Gegenstände of the Tractatus. The second leads to the identity of indiscernibles, and variables of quantification as the only referential terms in the language.[716]

Returning to the virtues of Descriptivism, it can apparently handle the problems faced by the Non-Descriptivist. So, if we consider again, 'The Morning Star is identical to the Evening Star', the Descriptivist theory precisely accounts for how such a statement can convey information even though the two names refer to the same object by positing that the *senses*, or meanings, of the names are different. Furthermore, as far as the statements 'Victor Jakob could not accept the new quantum physics' and 'Victor Jakob did not exist' are concerned, the Descriptivist view demonstrates how one can meaningfully use a name without supposing that there exists the entity so named. As Quine put it: "… where descriptions are concerned there is no longer any difficulty in affirming or denying being".[717] In this manner, a statement of existence or non-existence is not seen to contain any expression naming the entity whose existence may be in question. Thus, whether such a statement means anything

[715] Frege 1952, p. 61.

[716] Searle 1958, p. 169.

[717] Quine 1964, p. 7. See also Goodman 1951, p. 160 and Ayer 1954, p. 16.

or not does not presuppose that such an entity actually exists:

> We need no longer labor under the delusion that the meaningfulness of a statement containing a singular term presupposes an entity named by the term. A singular term need not name to be significant.[718]

Nevertheless, Descriptivism also faces difficulties. We've already noted that different people may attribute a different sense to the same proper name, as Frege himself realized.[719] Furthermore, there may be more than one possible description of the object which is available and it may be unclear to which of these the name should be associated. In particular, if contingent properties are included in such a description, then we run into the problem that any true statement about the object that used the name as subject would be analytic and any false statement would be self-contradictory. If, for example, the name 'Max Planck' is taken to mean 'the person who first regarded energy as quantized', then the statement 'Max Planck was the person who first regarded energy as quantized' becomes a tautology on this view. This is surely counter-intuitive, as the statement appears to express something which, with a little historical investigation, could be shown to be false.[720] And finally, the meaning of a name and perhaps the identity of the object would have to be regarded as changing every time there was a change in the properties of the object and the same name would have different meanings for different uses of the name.

Hence we have a dilemma: if we consider the role proper names play in language, then the Non-Descriptivist theory, in asserting that such names essentially refer but have no sense, seems to have it right; but if we think about the 'mechanism' of reference, then Descriptivism, in claiming that such names refer only if one and only one object satisfies the relevant description, seems to be correct. Again, as Searle puts it:

> The subject-predicate structure of the language suggests that the [Non-Descriptivist theory] must be right, but the way we use and teach the use of proper names suggests that it cannot be right: a philosophical problem.[721]

Searle's solution is to cash out the descriptive content associated with a name in terms of a sufficient but nevertheless unspecified number of 'standard

[718] Quine 1964, p. 9.

[719] Frege 1949, p. 86.

[720] Indeed, as we have noted, both Dorling and Kuhn have cast doubt on the truth of such a claim!

[721] Searle op. cit., p. 169.

identifying statements', where the use of the name to refer presupposes the existence of an object of which a sufficient but unspecified number of 'standard identifying statements' are true and to be that object is to possess a sufficient but unspecified number of the relevant properties mentioned in the 'standard identifying statements'. Of course, one might want to insist that certain of these statements carry more weight than others. However, the important point is that the lack of specificity in the number of 'standard identifying statements' is a function of the role played by proper names and to fully pin down these statements would be to reduce names to nothing more than descriptions and undermine this role. To ask for the full set of criteria for applying a name is to ask for the full set of identity criteria of the object. If such a full set of criteria were to be given, we would have a complete description of the object, and we could then only refer to it by describing it. But pragmatically, names don't function as descriptions like this but rather act as 'pegs on which to hang descriptions'.

This, Searle claims, resolves the problem: names do not have sense or meaning in the sense (ha!) that they are used to describe objects but nevertheless they are logically connected with the relevant characteristics of the objects to which they refer, in a 'loose sort of way'. Take the statement 'The Morning Star is identical to the Evening Star' one more time; then, Searle insists, this statement might well be analytic and uninformative for some people, since the same descriptive content may be associated with each name, but, of course, in many cases this content will be different and then the statement will be not only synthetic in these cases, but "might even advance a historical discovery of the first importance".[722] Or consider again the statement 'Viktor Jacob does not exist'; this is now to be understood as asserting that a sufficient number of descriptive statements associated with referring uses of the name 'Viktor Jacob' are false. Hence we can understand how the statement tells us more than simply that the name was never used to refer to any object; that is, we can understand how this statement is meaningful. Nevertheless, we can accommodate the points that different people may attribute a different sense to the same proper name and more than one possible description of the object may be available, without it being clear which should be associated with the name.

However, Searle's view also has unpalatable consequences. Let us take the name 'Albert Einstein' and the statement, attributed to Einstein, 'If only I had known, I should have become a watchmaker'. Is it possible for Einstein to have been a watchmaker? Intuitively, this certainly seems a possibility, one whose strength might be increased by the discovery that Einstein was handy

[722] op. cit., p. 173.

with tools, took watchmaking classes, etc. But on Searle's view, if Einstein had never been a physicist, he would not have possessed a crucial property mentioned in one of the 'standard identifying statements' which should surely carry more weight than perhaps any other; in other words, he would not have been Einstein. Searle himself bites the bullet here and insists that "… it is a necessary fact that [Einstein] has the logical sum inclusive disjunction, of properties commonly attributed to him …".[723] However, if we grant legitimacy to our modal intuitions on this and accept that it is possible for Einstein not to have been a physicist, then some account is required which allows us to refer to this person who was not a physicist but a watchmaker as 'Einstein'.[724]

Kripke has attempted to provide just such an account via the notion of a 'rigid designator' which designates the same object in every possible world, whereas a non-rigid or 'flaccid' designator, such as a description, does not.[725] Proper names are rigid because they satisfy this 'intuitive test'. As Kripke put it: "… although someone other than the U.S. President in 1970 might have been the U.S. President in 1970 (*e.g.* Humphrey might have) no one other than Nixon might have been Nixon"[726] and furthermore, "… although the man (Nixon) might not have been the President, it is not the case that he might not have been Nixon (though he might not have been called 'Nixon')".[727]

The central idea, then, is the following: we fix the reference of a proper name by an 'act of baptism', in which the object is named by ostension, or the reference is established by a description. The latter does not yield the meaning of the name, but simply fixes the reference via certain features of the object which the object has contingently, in this, the actual world. When this name is passed along the chain of communication, the person who receives it must then intend to use it with the same reference as the person from whom he or she heard it. Whether established by ostension or description, the name then rigidly designates the object which possesses this property in this world, in all possible worlds. Even if, in such possible worlds, the object no longer possesses the property used to fix the reference in the actual world, the proper name still rigidly designates that object. In other words, once its reference is fixed in the actual world, the designation of the proper names is unaffected by possible changes in the actual world which we can imagine.[728] Underlying this

[723] Searle 1969, p. 173; he actually uses the example of Aristotle.

[724] Here's another example, attributed to Kaplan: it is now accepted by some historians that although there really was a person who gave rise to the legend of Robin Hood, this person was not poor, did not live in Sherwood Forest, was not an outlaw, and his name wasn't even 'Robin Hood'!

[725] Kripke 1972, p. 48.

[726] Ibid., p. 49.

[727] Ibid.

[728] Ibid., p. 75.

view, of course, is a particular theory of the nature of possible worlds which holds that such worlds are not 'out there' as it were, to be viewed through some kind of metaphysical telescope, but are effectively constructed by us.[729] Thus, with the referent of 'Nixon' fixed in this world by, say, the property of being U.S. President in 1970, we construct a world in which Nixon does not possess that property but is a second-hand car salesman, say. Likewise, we may pick out the referent of 'Einstein' in the actual world as the physicist who proposed the theory of Special Relativity. In a possible world in which someone else achieved this result—Lorentz, say, or Poincaré—we would not refer to that someone else as 'Einstein'. If Einstein had never become a physicist, but had obtained gainful employment as a watchmaker, he would not have possessed this property which we use in the actual world to fix the reference of his name. This is precisely because the name 'Einstein' rigidly designates a certain person. Even though the property we have used to fix the reference is an accidental one, we still use the name to designate the person in all possible worlds.

Kripke's account is a version of the 'Causal' theory of proper names, so called because the original reference is recovered through a causal chain of communication by means of which someone using the name can be understood as intending to refer to that individual which was originally baptized with the name. According to this account, identity statements, for example, are, indeed, necessarily true, since the reference is the same, but they are not known *a priori* to be true, since the descriptions are different.[730] Thus we do not know *a priori* that the Morning Star is identical to the Evening Star, because the evidence by which we fix the referents of the names allows for the possibility that these referents might be different. The necessary truth of such identity statements ultimately expresses a relation between the object and itself—that is, self-identity—rather than a relation between names. Existential statements can also be dealt with: 'Victor Jakob does not exist' is not absurd, nor is 'Victor Jakob could not accept the new quantum physics' meaningless, since the relevant fiction or story in which the name appears substitutes for a 'baptism' and thereby supplies the meaning.[731]

[729] Of course, there need to be constraints imposed on such construction. A fairly weak constraint would be that of logical consistency, allowing any consistent description to be that of a possible world. Stronger constraints would then limit the set of such worlds. Thus, descriptions for which current physical laws hold would delimit the set of physically possible worlds. The weakest constraint, of course, would be none at all, so that 'anything goes'. We shall return to this issue of what goes into the construction of a possible world below.

[730] Kripke op. cit., p. 109.

[731] Donnellan 1974.

This view was proposed, at least in part, as a response to the problem of theory change noted above. It shifts focus from the apparently dramatic switch in theoretical descriptions across revolutionary divides, to the causal chains of communication tied to some original baptism event. The idea, then, is that even though, say, the theory of the electron has changed dramatically, we can still claim to refer to the electron through such changes as long as these causal chains are maintained. As it stands, however, this view still faces a problem of reference change. Consider the following example (from Hacking): in 1935 Yukawa proposed a theory of the nucleus which predicted the existence of a particle responsible for nuclear forces. This particle was named the 'meson'. In 1936 Anderson and Neddermeyer discovered a particle through observations of cosmic rays, which was identified with the meson. However, subsequent observations of the behaviour of this particle led to the conclusion that it was not Yukawa's meson at all and so it was re-dubbed the 'muon' (it was originally termed a 'mu-meson'). It was subsequently realized that the particle predicted by Yukawa was the 'pion', discovered in 1947.

What we have here is a case where a name is originally referred to one kind of particle, and is subsequently attached to another. When Anderson gave his particle the name 'meson', his intention was not, presumably, to introduce a novel use of the name, but to refer to whatever was referred to by Yukawa. But Yukawa intended the name to refer to the particle responsible for nuclear forces and this was not the same particle as Anderson's. One way that such reference change can be accommodated is via 'groundings' which occur after the initial baptism.[732] The idea is that a name comes to be 'grounded' in its bearer via subsequent observations which are semantically significant. Thus, observation of the behaviour of Anderson's particle led to a shift in reference of the name 'meson' and a re-dubbing of the particle as a 'muon'.

Nevertheless, this Causal theory remains open to the objection that it needs to be supplemented with a theory of meaning which can account for the fact that names do appear to have cognitive content. Thus, for example, there is still something rather unsatisfactory with its treatment of identity statements since the difference between 'The Morning Star is identical to the Evening Star' and 'The Morning Star is identical to the Morning Star' remains opaque. Yet this is the crucial difference that any account in which proper names themselves have reference but no meaning, seems incapable of explaining. Likewise, it is unclear how, precisely, the relevant fiction or story supplies meaning to

[732] Devitt 1981.

existential statements when the names themselves have no such meaning.[733] An obvious way of spelling this out would be in terms of the story providing an appropriate context but now there is the danger of this collapsing into a form of Descriptivism: of course, if we substitute 'the fictional protagonist of McCormach's novel who finds himself unable to cope with the changes in physics at the beginning of the twentieth century' for 'Victor Jakob', then it is clear that 'Victor Jakob does not exist' is only contingently true but the victory has been won at the price of conceding the explanation to Descriptivism. Furthermore, in order to fix the reference of a name, the dubber must at least know what *kind* of object it is. But that involves the satisfaction of a certain predicate and that means conceiving of the object in terms of a certain mode of classification, or simply, a description.[734]

These objections suggest that some descriptive element must be introduced into the account and hence various 'hybrid' theories have been proposed. Without going into too much detail, the general idea is that the reference of the name is now seen as the main causal source of the description which the speaker associates with the name.[735] Consider, for example, the case of a future (and possibly alien) historian of science who discovers a fragment of text in which the theory of Special Relativity is presented and which gives the name 'French' as the author of the text. The history of science community then uses the name 'French' to refer to the person who proposed the theory of Special Relativity, since their intention is not to introduce any novel use of the name and they believe that French did indeed propose the theory. However, it might be claimed that intuitively the name refers to the physicist formerly known as Einstein and not to the author of a well-known undergraduate textbook on Special Relativity. Hybrid theories handle this by insisting that it is the physicist who constitutes the 'dominant causal origin' of the description associated with the name, namely 'the person who proposed the theory of Special Relativity' and hence the name actually refers to that physicist.

It will be useful for our purposes to focus on Gracia's version of the hybrid approach, which explicitly and neatly incorporates the distinction between individuality and distinguishability.[736] According to this view, there are two aspects to the process of naming or 'tagging' an individual: the first concerns our epistemic access by means of which we become aware of the individual,

[733] Gracia 1988, p. 214.

[734] This is generally referred to as the '*qua* problem', since it suggests that in order to fix the reference of a name, the object must be thought of '*qua*' some kind.

[735] See Devitt op. cit., and Evans 1982.

[736] Gracia op. cit., Ch. 6.

the second is to do with the act of denoting, or giving the individual a name. The former involves description, whereas the latter "… has no descriptive mediation".[737] According to this approach, the primary function of proper names is to refer and the symbols employed refer not in virtue of any descriptive meaning they have but simply because they stand for the individuals concerned, where this individuality is understood in terms other than those of some set of properties, of course.[738] When this approach is applied to proper names,

> … we can see not only that proper names need not be descriptive in any sense in order to have referents, as descriptivists believed, but also that it makes sense to consider them primarily nondescriptive and therefore as devoid of meaning and sense.[739]

If we return to the issue of explaining *how* names refer, then from Gracia's perspective the concern here is simply misplaced. To seek a mechanism of reference is to misunderstand the primary function of names. Since it was this concern, this desire for an appropriate mechanism, which motivated the introduction of meanings via some form of descriptive element, there is now no need to hold that any such meaning is required in addition to reference.

When it comes to the issue of how names are *established*, however, we need to invoke some kind of act of baptism. What this does is to tie the name to a particular individual *as that individual* and not to a particular description, even in cases where the dubbing proceeds via description rather than ostension. This then allows for the possibility of changes, even fundamental changes, in the initial description.[740] On the other hand, it is through description that we *learn to use* these names effectively, in particular, when we are not acquainted with the bearer of the name. Thus, a description may play an epistemic role in distinguishing an individual from others, in terms of some set of properties, and in *helping* to fix reference through baptism, but it is not tied to the name by necessity.[741] As Gracia notes, this latter point can be taken as the essential insight of Kripke's discussion: we may use a description to get a fix on the bearer of 'Einstein', but once that fix is established we can change the components of the description through the construction of different possible worlds.

[737] Gracia op. cit., p. 219.

[738] Ibid., p. 217.

[739] Ibid.

[740] Ibid., p. 218. This applies also to natural kind descriptions (Gracia op. cit., p. 218). Thus the kind of particles which satisfy the initial description of instantiating the negative energy solutions of Dirac's equation and were originally regarded as protons, subsequently came to be classified as positrons.

[741] Ibid., p. 220.

Consider, yet again, the example of identity statements: on Gracia's account, the *function* of the names 'Morning Star' and 'Evening Star' is solely referential; in particular, there is no descriptive aspect to this function.[742] How, then, do such statements convey information? We learn to use the names 'Morning Star' and 'Evening Star' via descriptions involving both luminosity and spatio-temporal coordinates (for example, 'that bright object that appears at these positions in the sky on certain mornings/evenings of the year'). The information that we gain is that despite their differences, these are, in fact, descriptions of the same object.[743] That we are able to gain such information has nothing to do with the *referential* function of the names, but solely and appropriately with the *distinguishing* function of the descriptions. It is because we learn to use the names by means of this distinguishing function that we fall under the false impression that this function confers meaning upon the names; "[b]ut once it is understood that this is only a result of how we learned to use the names, then the difficulty vanishes".[744]

What about existential statements? Here again we must carefully separate the referential and distinguishing functions. Take the statement 'Max Planck exists'. The function of 'Max Planck' is to refer and we intend to do so when we invoke it. However, the reference may fail, of course. Consider the lessons of history: we learn to use the name via a description—such as 'the first person to quantize energy'—and, following the causal chain, we assume there was some baptismal event at the beginning of it. But it may turn out that there was no such event, that the name is an invention, say.

What fictional names refer to, Gracia suggests, are possible individuals.[745] Thus 'Victor Jakob' refers to such a possible individual and statements such as 'Victor Jakob does not exist' and 'Victor Jakob could not accept the new quantum physics' are not absurd or meaningless, even though the function of 'Victor Jakob' is to refer and not to take the place of a description.[746] Of course, what is then required is further elaboration of this notion of possible individuals but we shall not pursue this here. Again, whether such statements *mean* anything or not does not presuppose that the entities actually exist; the meaning is cashed out descriptively but the name is not taken to be *reducible* to this. Gracia's claim is that with this separation between referential

[742] Ibid., p. 222.

[743] In other words, and in this case, at least, these differences are deemed to be ultimately unimportant. What this corresponds to is the downgrading of certain properties in the description, such as those relating to the morning and evening appearances of the object.

[744] Ibid., pp. 222–3.

[745] Ibid., pp. 223–4.

[746] Ibid., p. 224.

and descriptive functions, 'Max Planck was the person who first regarded energy as quantized' is no longer tautologous and we can accommodate the possibility of different people associating different descriptions with the same name:

> … if … descriptions are only contingently related to individuality and serve to identify the bearer of the name only under very specific circumstances, then no problem arises by saying that proper names have no meaning and likewise for accounting for disparities of description.[747]

What this sort of account suggests is that the assertion that what names do is enable us to isolate and maintain contact with individuals[748] is too crude. The latter concerns distinguishability, rather than individuality itself, and here, as Gracia points out, a description may be invoked but is not necessarily tied to the individual.

With this discussion behind us, we can now address three broad issues concerning the role of names in physics. First, it has been claimed that naming practices in physics support some form of Descriptivist view rather than the Causal or 'hybrid' theories. Secondly, it has been argued that Kripke's theory of rigidly designating names and the construction of possible worlds is tied to *classical* statistics and finds itself in difficulties in the quantum domain. Thirdly, however, the non-classical indistinguishability of quantum particles presents problems for both the Descriptivist and Causal approaches.

This discussion will then set the scene for an outline of Dalla Chiara's and Toraldo di Francia's quaset theory which we shall present as a theory of quantum individuals for which distinguishing descriptions are unavailable.

## 5.2 NAMES AND THE PRACTICE OF PHYSICS

Given that scientific practice is primarily concerned with distinguishability and re-identification, it should come as no surprise that physicists appear to name and refer to objects in terms of a cluster of descriptive properties. This may suggest that support for the Descriptivist theory of names can be drawn from scientific practice but, of course, a certain degree of metaphysical care must be taken at this point. *If* individuality is collapsed into distinguishability, then this suggestion might appear reasonable. On the other hand,

[747] Gracia op. cit., p. 219.

[748] McCulloch 1989.

if one sought to maintain this distinction, then one might insist that such practice bears on issues of distinguishability only and the Causal or hybrid views could be retained in the face of practice (of course it may do nothing to resolve the philosophical difficulties these views face). In other words, scientific practice itself does not determine which view of proper names should be adopted.

This is important because it might appear that this practice *does*, in fact, bear unequivocally on this issue. Thus, Dalla Chiara and Toraldo di Francia consider the following interesting case taken from astrophysics:[749] two quasars are discovered lying very close together and are initially labelled differently, by Q1 and Q2, say. It is then suggested that 'in fact' these are just two images of the same astrophysical object, whose light has been refracted by a 'gravitational lens'. As Dalla Chiara and Toraldo di Francia point out, this example is not quite comparable with that of the Morning Star and Evening Star above, since Q1 and Q2 are observed at the same time. Hence, they claim the names cannot be simply dispensed with. In other words, the underlying metaphysical issue is not that of re-identification through time, which is what the identity of the Morning Star and Evening Star effects, but, rather, that of distinguishability and individuality.

What are our options here? Well, as Dalla Chiara and Toraldo di Francia note, one could insist that the referents of the names are just the spots of light themselves, so that the statement 'Q1 is identical to Q2' is actually rendered as strictly false. Although they do not pursue this line themselves, such a move has obvious positivistic overtones and even the anti-realist will, these days, accept that names can be used to designate unobservable objects lying beyond the appearances. In this case, the names designate the stellar object giving rise to the two spots of light. Now, however, Dalla Chiara and Toraldo di Francia insist that "… when we use names to designate *stellar objects* instead of light spots, we use them as abbreviated descriptions".[750] Of course, the description will incorporate the cluster of properties deemed to be 'essential' by the astrophysical community; these will be properties such as luminosity and not, most importantly, the positions in the sky of the separate light spots. Dalla Chiara and Toraldo di Francia take this to show that "… even in macrophysics, the legitimacy of a sharp distinction between *true proper names* [presumably in the sense of the Causal or 'hybrid' theories] and descriptions may be highly questionable".[751]

[749] See their 1993 paper.

[750] Ibid., p. 263.

[751] Ibid., p. 264.

Searle has drawn on the example of storms and hurricanes to make a similar point: these are—famously—given names by scientists on a descriptive basis, where the description crucially involves a specification of location in terms of latitude and longitude and is thus ultimately spatio-temporal.[752] But, of course, these claims for the role of descriptions in physics are contentious. In particular, the practices that are drawn upon in these examples may appear to undermine the distinction between the Causal and Descriptivist approaches only if one has given up on, or failed to make, the prior metaphysical distinction between distinguishability and individuality. That scientists use descriptions for identification says nothing about whether the 'true proper name' of the quasar or storm can be *reduced* to such descriptions. As Gracia emphasizes,

> … even though the scientist may know the individual only through some kind of description, when he imposes the name, he tags the individual. There are then two aspects to be considered in this process: (1) the epistemic paraphernalia through which the scientist becomes aware of the individual out there—and all of this involves description—and (2) the act of giving the individual a name—and the latter connection has no descriptive mediation.[753]

Thus, it is certainly not the case that either the Causal or 'hybrid' theories are undermined by such examples. However, an argument has been put forward to the effect that Kripke's view is ruled out by quantum physics. In the next section we shall consider this claim and as we shall see, the argument is not entirely straightforward.

## 5.3 NAMES, POSSIBLE WORLDS AND PARTICLE STATISTICS

As we have indicated, by initially introducing sub/super-scripts in our description of an assembly of quantum particles, it may appear that we are labelling and hence naming them. The problem, of course, is that applying a permutation shuffles these names between the particles in such a way that we cannot tell which label 'belongs' to which particle. Let us begin with the claim that all that names do is refer to individuals and they have no meaning over and above that reference. How does this claim fare in the quantum context? Here is an argument that suggests it fares rather badly.

[752] Searle 1984, p. 241.

[753] Gracia op. cit., p. 219.

In *Naming and Necessity*, Kripke introduces his account of rigid designation and possible worlds by way of an elementary example from probability theory. He asks us to imagine throwing a pair of dice and considering the possibilities for the way the dice may land.

> [No] school pupil [would] receive high marks for the question 'How do we know, in the state where die A is six and die B is five, whether it is die A or die B which is six? Don't we need a "criterion of transstate identity" to identify the die with a six—not the die with a five—with our die A?' The answer is, of course, that the state (die A, 6; die B, 5) is *given* as such (and distinguished from the state (die B, 6; die A, 5)). The demand for some further "criterion of transstate identity" is so confused that no competent schoolchild would be so perversely philosophical as to make it. The 'possibilities' simply are not given purely qualitatively (as in: one die, 6, the other, 5). If they had been, there would have been just twenty-one distinct possibilities, not thirty-six.[754]

Thus he insists that the above possibilities are just abstract states and not 'complex physical entities' composed of 'phantom' counterparts (as in Lewis's account of possible worlds), by noting the absurdity in asking for criteria of 'transstate' identity to identify which of the dice in an abstract state are identical with one of the actual die. Such states—possible worlds—are stipulated, rather than 'viewed from afar'. However, as Maidens has pointed out, Kripke has just stipulated his way into a commitment to classical Maxwell-Boltzmann statistics![755]

Let's consider the above passage in more detail. Maidens suggests that the Kripkean account offers us two things.[756] Firstly, the rigid designators "A" and "B" enable us to stipulate worlds as different one from another. Second, the fact that the worlds *can* be so stipulated explains the observed statistics. Putting these together, we have an argument that rigid designation leads to classical statistics and the failure of such statistics, given QM, leads to the possibility of a form of modus tollens against rigid designators.[757] Thus we appear to have a case where the physics can rule out a particular metaphysical view of proper names. As we shall now try to spell out, the case is not quite as cut and dried as it might seem.

[754] Kripke 1972, p. 17.

[755] Maidens pre-print.

[756] Maidens 1998.

[757] There is an obvious comparison here with Redhead and Teller's claim that quantum particles should not be considered as the bearers of names (Redhead and Teller 1991): if they could bear names, we could stipulate possible worlds represented by non-symmetric states; since such states are never observed, our stipulation has gone awry; the only way of avoiding such stipulation is to acknowledge that quantum particles are not the bearers of names. As we shall shortly see, this is *not* the only way to avoid the offensive stipulation.

Maidens draws on the work of Dalla Chiara in order to articulate the above argument.[758] Thus she notes that if we consider a single particle in a single state, such as an electron in a state of spin 'up', it seems natural to read the formalism in subject-predicate form: $|a_1^1\rangle$ translates as 'the appropriately labelled or named particle has the property represented by being in the appropriately labelled state'. Standardly, we would then assign appropriate denotata to the names and appropriate extensional interpretations to the predicates. We would also, standardly, invoke the Fregean 'Principle of Compositionality' which holds that the truth-value of expressions such as the above is 'composed' of the truth functions of the separate parts; in other words, in order to determine whether the expression is true or false, we need to be able to determine whether it is true or false that the object has the property predicated of it and, crucially, whether the object has been correctly named in the first place. As Dalla Chiara puts it, "… one has first to identify the *denotatum* of the name …".[759] But it is precisely this latter identification which is problematic in QM, and thus compositionality fails, or so it is claimed.

Consider the superposition formed by a pair of electrons and the spin states 'up' and 'down': $\frac{1}{\sqrt{2}}(|a_1^1\rangle|a_2^2\rangle - |a_2^1\rangle|a_1^2\rangle)$. In this case, we can assert that 'there is one electron which has spin up', but not '*this* individual has spin up': there does not appear to be any way of 'picking out' the individual which has one name, rather than the other.[760] The conclusion is that,

> There is no trans-world identity. In this situation the meaning of "rigid designator" becomes very fuzzy. Anyway, the term seems useless".[761]

Returning to the Kripkean example, if we try to think of a possible world containing only such a superposed state of two particles it is hard to see how swapping the labels would be equivalent to stipulating a new possible world. Swapping labels in this case seems to merely re-describe the same world and suggests that the labels are in this case not functioning as rigid designators in Kripke's sense.

However, this conclusion can be resisted. We shall offer a detailed account of Dalla Chiara's and Toraldo di Francia's approach below, but let us consider first of all how the above situation differs from a classical example. Imagine a world containing two indistinguishable classical particles which are individuals and suppose the description of this world only has rigidly designating names for

[758] Maidens ibid., pp. 83–4.

[759] Dalla Chiara 1985, p. 189.

[760] Maidens op. cit., p. 84.

[761] Dalla Chiara and Toraldo di Francia 1993, p. 267; cited in Maidens op. cit., p. 84.

these indistinguishable individuals. Swapping the names in such a description also appears to generate a description of the same possible world, but here the names were rigid designators by hypothesis and hence such examples cannot count against Kripke's notion. However, Maidens, Dalla Chiara and Toraldo di Francia might point out that in the classical case the particles could be 'picked out' in principle, whereas in the quantum situation they cannot, because they are in a superposition. Therefore, appearances to the contrary, swapping the labels in the classical case should be taken as generating a different possible world and the asymmetry between this and the quantum case remains. Of course, one can question this assertion that the particles can be picked out in principle, particularly since it is precisely such symmetric worlds which have been presented as counter-examples to the Principle of Identity of Indiscernibles.

We recall, however, that at least one form of the Principle, namely PII(1), can be sustained in the classical domain, and if we are willing to accept some form of substantivalist view of space-time, such that the particles can be said to possess different spatio-temporal properties even in this symmetric world, then it would seem we should conclude that swapping the labels leads to a different possible world. With PII violated in the quantum case, this option is ruled out (and even Saunders' version of the Principle does not allow us to say which particle has which label). Thus the issue depends on our understanding of such symmetric worlds and it all comes down to whether the above world containing a quantum superposition is sufficiently different from a classical symmetric world, such that in the former, but not the latter, a permutation of the particle labels cannot be said to yield a different possible world. If it is not sufficiently different—at least not in the relevant sense—then the conclusion that the particle labels cannot act as rigid designators is problematic.

Secondly, consider the claim that the Principle of Compositionality requires us to first *identify* the denotatum of a name. Our concern is that this claim implicitly conflates distinguishability with individuality. What compositionality is about is the *epistemological* issue of ascertaining the truth value of, typically, subject-predicate sentences. In order to do so in the cases considered, one clearly has to *distinguish* the individuals concerned but the fact that one cannot do this in the case of the superposition above does not imply either that one cannot regard the objects as individuals or that they cannot be named. The epistemology does not dictate the metaphysics.

Of course, the metaphysics may be regarded as bizarre, as we have already noted. What would have to be claimed is that we have a situation involving

named individuals but we cannot, even in principle, say which individual has which name. In effect, the superposition hides or 'veils' the individual objects. Is there a theory of names which can accommodate such a bizarre situation? What about the Descriptivist view of names, or particle labels, according to which they are regarded as reducible to definite descriptions? Maidens claims that "… insofar as particulars can be named, it is in virtue of their possessing qualitative thisness",[762] where 'thisness' is taken to be, not irreducible, but dependent on the properties and relations of things. However, quantum particles cannot possess such qualitative thisnesses, because of the entanglement represented by superpositions such as the above (that is, because we cannot say which electron has which property; ibid.). In other words, as Maidens points out, the Descriptivist approach won't work, since the descriptions in this case will be identical (and we recall that this is why the Principle of Identity of Indiscernibles fails, in even its weakest form). Nevertheless, the descriptivist can at least *explain* why: entities in such superposition states just don't admit of uniquely distinguishing descriptions. Maidens' conclusion is worth quoting in full:

> If we accept that absence of qualitative thisness explains the problems we see with failure to be able to attach names to these particles in any way which enables us to pick out the objects denoted, we have a powerful argument for Lewis's claim that representation *de re* supervenes on qualitative characteristics. Names may function rigidly, but this is in virtue of features of the world which metaphysics and physics can help us to examine … It is not that a consideration of the workings of names within our language alone suffices to enable us to draw metaphysical conclusions.[763]

Here the view is made explicit that the referential function of names is to be understood solely in terms of 'picking out' the objects named. And on this understanding, the labels attached to quantum particles would indeed be meaningless. But absence of qualitative thisness—which is just another way of phrasing the non-classical indistinguishability of quantum particles—need not imply a lack of such referential function, as we indicated in the introduction to this chapter. To this extent, the quantum case need not be seen as providing any sort of argument for Lewis's claim regarding *de re* representation; indeed, one might regard such a claim as begging the question at issue.

Clearly, then, any view which allows the labels to have a referential function, even in situations of descriptive equivalence, will restore their meaning. Let's consider some of the alternatives.

[762] Maidens 1998, p. 85.

[763] Ibid., pp. 85–6.

As we have just indicated, and as we recall from Chapter 4 it is the apparent identity of descriptions in the above cases which provokes the failure of PII. However, as we also noted, this failure has been resisted by van Fraassen, for example, through an appeal to 'empirically superfluous' factors which serve to distinguish and hence individuate the particles. Could such factors be regarded descriptively as forms of names? The nature of the factors involved is dependent on the statistics. Fermions are regarded as individuated by their 'value' states which, unlike the 'dynamic' states, need not be the same for every member of an aggregate. We can assign different value states to distinct particles, in the sense that for any two particles, some observable has different values (and not that there is a single observable with a different value for each particle).[764] With bosons, however, the same dynamic *and* value states will be assigned to all members of the aggregate and so genidentity is appealed to, where this is regarded not only as 'empirically superfluous' but also, unlike the fermion case, not describable in quantum mechanical terms.[765] The difference, then, is that with fermions, the individuating factor is describable in quantum mechanical but not empirical terms, whereas with bosons it is describable neither theoretically nor empirically.

This gives a two-fold view of particle labels, depending on the statistics:[766] for fermions, the labels can be understood in descriptive terms, where the description is empirically inaccessible; for bosons, the labels cannot be reduced in this way, unless the description is taken to include spatio-temporal factors. This may seem metaphysically uneconomical,[767] particularly as the particles are

[764] van Fraassen 1991, pp. 427–8.

[765] Ibid., pp. 429–33. van Fraassen appeals to the work of Mittelstaedt as showing that "… re-identification over time has no empirical significance in quantum mechanics" (ibid., p. 430) but notes that Mittelstaedt understands naming in descriptive terms only. He also cites Dalla Chiara and Toraldo di Francia in the same context but resists their move to quasets precisely because he feels he is able to give an account of particles in which they are regarded as named individuals. The problem, as we shall indicate, is that, as with the other accounts of individuality which appeal to factors over and above the empirically describable properties, van Fraassen must still face the sort of situation that motivates the move to quaset theory. In other words, he must still face situations in which we have named individuals but for which we cannot say which individual has which name and it is precisely such situations which, we maintain, must be described in terms of quasets (and as we shall emphasize, it is this which marks the difference with quasi-sets, which describe non-individual entities).

[766] Although van Fraassen acknowledges the possibility of parastatistics (1991, pp. 399–400), he does not consider the issue whether paraparticles can be individuated. Given the mixed symmetry of paraparticle states, there may be problems here for his view and the question arises whether the dynamical and value states are distinct or not in such states.

[767] van Fraassen (1984) briefly considers the issue of naming in the quantum context, citing, again, the work of Mittelstaedt and Dalla Chiara and Toraldo di Francia. As formal possibilities, he cites his own work on abstractors (1982), which, interestingly, is also cited by Dalla Chiara 1985 and a general account of quantifiers proposed by Daniels. This second proposal bases quantifiers on functions which take propositions to numbers and, as van Fraassen notes, this is suggestive of the idea that even though the particles are (strongly) indistinguishable, *number* remains a well-defined observable (1999, pp. 89–90). These suggestions, regrettably, are not taken up in van Fraassen 1991. The second, in particular, may be related to the work on qua- and quasi-sets we shall discuss below.

still regarded as indistinguishable, of course, and so the gap remains between the metaphysics of naming and the epistemology of distinguishing. As we shall note, this gap is faced by all such accounts which appeal to factors over and above those that can be empirically described. Furthermore, it is our contention that, because of this gap, a non-standard semantics is required, even if the particles are still regarded as individuals.

We may thus save a form of the Descriptivist account, but what about the Causal theory of names? This faces the difficulty that the crucial component of this account, namely the existence of some kind of causal chain connecting the use of a name with its referent, does not appear to be supported by quantum mechanics. As van Fraassen points out,

> [h]owever obscure the notion of causal chain may be, this view of reference would seem to preclude differential naming of two photons in the same state—since entering into distinct causal chains would surely distinguish them in a way that quantum mechanics does not recognize.[768]

There is also an issue concerning baptism: on this view, we recall, the referent of a name is established either by ostension or via some distinguishing description. However, ostension is clearly ruled out and descriptions are problematic in QM, as we have just seen. One might argue that in a measurement—when an electron hits a scintillation screen, for example—we can provide such a description using spatio-temporal properties, but of course the ability of such a description to distinguish is only fleeting: as soon as the electron enters into a superposition once again it is gone.[769] Nevertheless, one might want to press the case here and insist that since it is precisely in such measurement situations that we are led to conceive of the particles as individual objects to begin with, there is a kind of 'pseudo-' or 'mock' baptism, on the basis of which names can be ascribed. Of course, immediately outside such situations the possibility of keeping track of which particle has which name or label is lost and, as we have seen, there can be no causal chain.

[768] 1999, p. 89. "Unless [he continues] … the notion of causality be sufficiently metaphysical, or the quantum mechanical description of nature sufficiently incomplete!" (ibid.). Of course, the causal theorist may wish to argue for a notion of causality that includes an empirically superfluous component.

[769] Here we have a kind of 'mock individuality' again; we shall return to this below.

Alternatively, we might combine aspects of the Causal and hybrid views to yield, again, a form of baptism. We recall Donnellan's suggestion that when it comes to fictional names, the relevant story in which the name appears substitutes for a 'baptism'.[770] Perhaps the theoretical context of QM can serve a similar function, particularly if we follow Gracia's point that in such a 'baptism' the name is associated with a particular individual as that individual and not with a particular description. We can then, perhaps, entertain a kind of 'theoretical baptism' in which we assign names to individuals whilst setting up the quantum formalism. That we then lose the possibility of saying which name is tied to which individual after we construct the state-function of the assembly is strictly irrelevant to *this* aspect of naming. What it is relevant to is the distinguishing aspect, of course, but this is invoked, on Gracia's view, only with regard to how we *learn to use* names. Since we are not concerned with this in physics—we don't use *names* in the practices of either classical or quantum physics, but only descriptions (we shall return to this point below)—there are no obstacles to using names to refer and to baptize. This may seem only the shallowest of baptism ceremonies but unfortunately it's the best we can get in this context.

Of course, in all these cases where names are granted some referential function, the gap between metaphysics and epistemology yawns wide. It is at this point that one may feel that the former should be more closely tied to the latter; this is a legitimate feeling, our point is simply that we are not driven by the physics to give up this particular package of named individuals.

Returning to the passage from Kripke, what about the point that he has tied his act of stipulation too closely to classical statistics? There is a straightforward response to this.[771] As we mentioned above, constraints may be imposed on what we can imagine when it comes to possible worlds. We recall the example given in the Leibniz-Clarke correspondence, when Leibniz asserts that although we can *imagine* a situation in which there are two indistinguishable individuals, such a situation is not a *genuine* possibility being contrary to God's will. A similar tack might be taken on stipulation; indeed, as we have already noted, Hacking makes a similar move in his attempt to save PII from the standard Blackian two-globe counter-examples: "In arguing that in a certain possible world there exist two distinct but indistinguishable objects, bland assertion is not enough. There must be argument".[772] Thus, when it comes to Kripke's example, we might insist that

[770] Donnellan 1974.

[771] French 1998.

[772] Hacking 1975.

'cold naked stipulation'[773] is not enough and that we must apply a form of stipulation dressed up with what we have discovered about the actual world in the form of quantum statistics; that is, the answer in the probability example is *not* given *a priori* but only on the basis of our understanding of what the actual world is like.

Of course, an obvious concern here is that invocation of the actual world in this manner smacks of the view (associated with Lewis) in which assessments are made of the 'closeness' to the actual world of possible worlds containing counterparts to individuals in the actual world.[774] But giving an account of which possible worlds are physically possible in this way still isn't *discovery*. What is essential and attractive about Kripke's approach is the idea that possible worlds are stipulated, rather than 'viewed from afar'. Where he went wrong was in choosing the analogy of a pair of dice and their possible states and then claiming that the latter are just 'given as such'. In a sense they are, for objects which obey Maxwell-Boltzmann statistics; for other kinds of objects the set of permissible states will be different and fully clothed stipulation is required to ensure we end up with the 'right' set.[775] What is important is to avoid having to specify criteria of counterparthood and we see nothing here that compels us to do that. Kripke himself invoked an aspect of the actual world in drawing on the dice example; his mistake lay in failing to note the other possibilities that exist in this context.[776]

There is one final problem to be dealt with. We recall the point that it is the stipulation of possible worlds based on rigid designation which is taken to explain the statistics of dice. If we are attempting to save rigid designation by insisting on constrained stipulation, then we appear to have lost the possibility of explaining the statistics. Indeed, the relationship has now been reversed, with the appropriate statistics feeding into the constraints on possibility. Can we find an alternative explanation? Not surprisingly, we think we can, but it means giving up on the approach which insists that the explanation of the statistics of possible outcomes lies in counting possible worlds, where one makes note of those which are qualitatively indistinguishable yet distinct.[777] According to the particles-as-individuals package, quantal particles may be regarded as (named) individuals subject to certain initial conditions which

[773] Hacking ibid.

[774] Maidens forthcoming, pp. 16–17.

[775] Cf. Dorling's remark: "You can't get out the statistics without putting in the dynamics …" (Dorling 1978).

[776] Maidens herself has hinted at something similar when she invokes the time honoured analogy between quantum particles and pounds in a bank account.

[777] Maidens pre-print; Toraldo di Francia 1985.

impose restrictions on the set of states accessible to them. It is *this* which accounts for the statistics. Of course, an explanation might be demanded of the initial conditions themselves. At this point, as we indicated in the previous chapter, one could simply dig one's heels in and insist that this is simply the way the world is. This alternative does at least have the virtue that it meshes with our formal practice, where we begin by labelling the particles and states.[778] We should, perhaps, re-emphasize what we have said before: we are not insisting on the particles-as-individuals package but at least it is not ruled out as a metaphysically viable alternative.

## 5.4 NAMES AND NOMOLOGICAL OBJECTS

As we noted above, Dalla Chiara and Toraldo di Francia suggest that the practice of physics supports a Descriptivist view of names in general. When it comes to QM they insist there are no proper names to begin with, and hence quantum particles are 'anonymous'.[779] Two important questions then arise: why are there no proper names at the microphysical level? And how can we talk about what happens at this level if there are no names?

With regard to the first question, Dalla Chiara and Toraldo di Francia's answer is that "[t]he lack of proper names is due fundamentally to the fact that the objects of microphysics are *nomological*".[780] By 'nomological' they mean that not only do particles of the same kind possess the same set of state-independent or intrinsic properties, but that these properties are 'fixed' by physical law. Now, at first sight this seems a little curious, since one might think that *classical* particles are also nomological in this sense. Before pursuing Dalla Chiara and Toraldo di Francia's account of names in QM, we shall digress a little in order to explicate this notion of 'nomological' objects.

Classical objects, Toraldo di Francia maintains, are not 'nomological', since "… their individual configuration had nothing to do with laws".[781] The idea appears to be that in classical mechanics the value of the *mass* of a particle, say, is not given by law, so that it can have a contingent range of values.[782] For a nomological object, on the other hand, this value, and also those of other properties such as charge, angular momentum etc., is well determined

[778] Maidens suggests that Dalla Chiara's work casts doubt on this alternative (Maidens 1998, p. 84). However, as we have indicated and as we shall further discuss below, this work fails to make the distinction between individuality and distinguishability that this alternative insists upon.

[779] Ibid., p. 264.

[780] Ibid., p. 265.

[781] Toraldo di Francia 1978, p. 62.

[782] Toraldo di Francia ties this in with the development of field theories and continuum physics in general.

and 'prescribed' by physical law.[783] The crucial 'discovery' appears to be that of discreteness, in the sense that the value of a particle's charge can be expressed as an integral multiple of the charge of an electron, angular momentum is quantized as multiples of $\hbar/2$ and so on.[784] In this sense, then, the values of a property cannot take any of a continuous range of values. And, of course this has been seen as the re-emergence of a fundamental form of atomism in physics, in terms of which one might assert that the notion of physical object has been 'recovered'.

However, it is still not clear in what sense these properties are 'well determined' and the objects themselves 'prescribed' by law. After all, the charge of an electron, Planck's constant etc., all represent fundamental constants whose values cannot be obtained by physical law; indeed, attempts to so obtain them (the most famous, perhaps, being Eddington's) are typically dismissed as mere numerology. What Toraldo di Francia seems to have in mind is a broader conception of what counts as a physical law: for instance, one can formulate the law that a mass $m = 9{,}1 \times 10^{-23} g$ must always be accompanied by an electric charge $e = \pm 4{,}8 \times 10^{-10}$ *e.s.u.*, by a spin $\hbar/2$, and so on.[785] But as he acknowledges, with this conception all the objects of physics are 'more or less' nomological, including, for example, planets with rings such as Saturn and Uranus (ibid.). What we can perhaps discern here is the suggestion that the individual is submerged under the kind in modern physics, in the sense that an electron, for example, is defined to be that kind of thing which has a mass of $9{,}1 \times 10^{-23} g$ and a charge of $4{,}8 \times 10^{-10}$ *e.s.u.* and so on, so that anything which possesses this set of properties has to be an electron. Furthermore, the discreteness of these values implies that such sets—characterizing muons, quarks and so on—are sharply delineated. Thus, the view of objects as 'nomological' ultimately reduces to a form of bundle theory, as Toraldo di Francia himself acknowledges: "*In some way*, physical objects are today *knots of properties*, prescribed by physical laws".[786]

The 'recovery' of integral numbers inherent in the above discreteness appears to reduce measurement to counting.[787] But the counting of elementary particles is problematic and here Toraldo di Francia raises concerns about individuality:[788]

> Can we distinguish *this* and the *other*, in a system of two electrons? As is well known, this cannot be done: identical particles are indistinguishable. Here, cardinal numbers

[783] Toraldo di Francia 1978, p. 63.

[784] Ibid., p. 64.

[785] Ibid., p. 63.

[786] Ibid., p. 63.

[787] Ibid., p. 64.

[788] Ibid., p. 64. He actually devotes more space to the issue of the nature of virtual particles, concluding that they are "… much fuzzier and more elusive … than anything we would include in the common sense notion [of object]".

> seem to take over the role we had previously attributed to ordinal numbers. A system of identical particles has a *cardinality*; but we cannot tell which is the first, the second, and so on."

Setting aside the conflation of individuality with distinguishability, the conclusion that an assembly of quantum particles may have cardinality but not ordinality has an impact on the foundations of logic:[789]

> Think, for instance, that in any formalised theory, we must start by defining a *universe of discourse*, that is a set of objects we want to talk about. For this to make sense, it must be well determined whether an object belongs or not to the set (*extension* of the set); moreover, the objects must be distinguishable from one another, so we can tell which is which. These requirements can perhaps be met by mathematical objects (although some scholars strongly oppose the idea). But what happens in the case when our universe of discourse is made up of physical objects? How can we tell that it is determined whether a given electron belongs or not to our system, and how can we tell which electron we are talking about?"

The alternatives Toraldo di Francia proposes are to employ fuzzy set theory[790] or develop an intensional semantics appropriate for such objects.

This leads us to quaset theory, but before we return from our digression, we wish to note a number of points concerning the notion of 'nomological objects' and the way it feeds into subsequent considerations of denotation and set theory: first of all, the underlying metaphysics appears to be one of (non-classically) indistinguishable *individuals*, where the individuality of the particles—*as objects*—is understood in terms of a form of bundle theory, but without PII, of course. Secondly, there appears to be something of a *lacuna* between Toraldo di Francia's exploration of the notion of nomological objects and the above conclusion concerning the foundations of logic. Indeed, the latter appears to follow solely from the previous consideration of the nature of indistinguishability in QM.

The situation is muddied somewhat by Dalla Chiara and Toraldo di Francia's subsequent remarks on this issue. As we have already noted, they insist (1993) that the lack of proper names in QM is due to the fact that the particles are nomological objects. There they explicate this notion both in the terms given by Toraldo di Francia—that is, of objects whose characteristics are 'fixed' by physical law—*and further*, in terms of what they now refer to as the 'identity' of electrons. Indeed, in remarks that recall Weyl's comments

[789] Toraldo di Francia 1978, p. 65.

[790] And here he cites the example of virtual particles again.

about the electrons' lack of an alibi, they ask, "How can you talk of Peter or Paul if you cannot distinguish the one from the other …?".[791] Thus, it appears that it follows from the notion of a nomological object that two or more such objects are 'absolutely indistinguishable'. But if this were the case then Saturn or Uranus could not be nomological. Perhaps we could clarify the situation if we were to characterize their position in the following way: all objects in physics are 'more or less' nomological, in the sense that the metaphysical underpinning of substance has been removed, to be replaced by that of invariants understood in terms of fundamental properties. In this sense, particles are just 'knots' or bundles of such properties. Quantal particles are objects in this sense which are, further, absolutely indistinguishable (and which violate the Identity of Indiscernibles). Thirdly, and in particular, the crucial concern arising from this metaphysics is whether or not the objects in question can be said to *belong* to a set, in the standard sense. As we shall see, it is on this point that *qua*-set theory can be viewed as distinct from *quasi*-set theory.

Let us now consider Dalla Chiara and Toraldo di Francia's second question: if quantal particles are anonymous, how can we talk about them? As we have seen, they consider the notion of a rigid designator to be a dubious one even in the domain of 'macrophysics'. In the microphysical realm it is rejected as 'useless', because it is claimed absolute indistinguishability implies that there is no trans-world identity.[792] Of course, one might object that there are situations in which the overlap of the wave functions of two electrons, say, is sufficiently small that we can treat the particles as if they were distinguishable and assign proper names to them. It is this that Toraldo di Francia refers to as a 'mock individuality' and writes,

> This is why an engineer, when discussing a drawing, can *temporarily* make an exception to the anonymity principle and say: "Electron *a*, issued from point S, will hit the screen at P, while electron *b*, issued from T, will land at Q".[793]

However, he insists, these names cannot be regarded as rigid designators and cannot denote the same object in all possible worlds.[794] Granted that an electron could be sufficiently isolated that we could—temporarily—distinguish it and regard it as named, if we were to shift to a possible world in which the electron

[791] op. cit., p. 265; see also Toraldo di Francia 1985, p. 207; note, again, that the question is posed in terms of distinguishability.

[792] op. cit., p. 267; Toraldo di Francia 1985, p. 210.

[793] Toraldo di Francia 1985, p. 209; Dalla Chiara and Toraldo di Francia op. cit., p. 266.

[794] Subsequently, this mock individuality is referred to as "conventional", and, it is insisted, "… has nothing to do with *substantiality*" (Toraldo di Francia 1985, p. 211; Dalla Chiara and Toraldo di Francia 1993, p. 269; their emphasis).

interacts with others, giving rise to a superposition, the distinguishability, and hence the possibility of naming, would be lost. Thus the identity of the electron cannot be maintained across all possible worlds. Note that the basis of this claim—the problematic nature of superpositions—also supports the claim that there is no temporal identity for electrons (since if an electron, again sufficiently isolated as to allow it to be temporarily and approximately distinguished, interacts with another electron in this, the actual world, we cannot tell, even in principle, which electron is which after the interaction). In other words, it is not the re-identification of the electron across possible worlds *per se* that is the problem, but the re-identification of the electron after entering a superposition.

Mittelstaedt has suggested that we can still speak of a 'weak kind' of trans-world identity in this sort of case if we take names to denote not single objects but classes of indistinguishable systems.[795] With such classes defined in terms of the relevant essential properties of the particles, we might regard them as classes of nomological objects in Toraldo di Francia's sense. If the system in the actual world is prepared in a given state, then those possible worlds which are 'accessible' to the system are defined to be those for which the system can be in states related quantum mechanically to the given state (cf. our previous discussion of 'accessibility', where the symmetry of the Hamiltonian guarantees that for a system in a state of a given symmetry, only those states of the same symmetry will be accessible to the system). Since this relation is typically probabilistic, Mittelstaedt argues that the trans-world identity of such quantum systems can be approximately preserved.[796] Of course, as we have noted, the trans-world identity here is not of a single object, but of a whole class or kind of objects. Thus the semantics being suggested is only Kripke-*like*, not only with regard to the role of probability in QM[797] but also with regard to the nature of the denotata.

With rigidity deemed inappropriate for describing single quantum mechanical objects, Dalla Chiara also offers a kind of 'liberalized' Kripke semantics, but one which appears to go much further than Mittelstaedt's suggestion.[798] One way of entering this formalism is via a comparison with so-called Kripke

[795] Mittelstaedt 1985.

[796] Ibid., pp. 230–1. Likewise, he argues that trans-temporal identity can be only approximately preserved, through the employment of 'unsharp' position measurements and probabilistic evolution of states (ibid., pp. 225–6); cf. Toraldo di Francia's 'mock individuality'.

[797] Ibid., p. 230.

[798] Dalla Chiara 1985. The following passages are taken from a discussion with Anna Maidens; we are grateful to Anna for allowing us to reproduce this informal joint work.

models.[799] Consider a language $L$ with individual names $a_m$, predicates $p^n_m$, variables $\varkappa_m$, and logical constants ¬, ∧ and ∀. Following Dalla Chiara, we introduce a set of possible worlds $I$ and an accessibility relation ± between them which is symmetric and reflexive. Sets of worlds $X$ are possible propositions containing all and only worlds whose accessible worlds are not inaccessible to the whole set. We can then construct sets of possible propositions $P$ which are closed under the operations of orthocomplement and union, and contain ∅ and $I$. A Kripke model is a system $\mathcal{M} = \langle I, \pm, P^U, D, r \rangle$ *with* $U$ some family of subsets of $P^U$ closed under infinitary intersection, where the important features for our purposes are that: (1) For any world $i$, the set $D$ represents the domain of $i$; (2) $r$ gives extensional meanings to non-logical constants—so $r_i(a_m)$ (the denotatum of $a_m$ in $i$) is an individual in $i$ and $r(p^n_m)$ (the extension of $p^n_m$ in $i$) is an $n$-ary relation on $D$. Names are rigid in that $r_i(a_m) = r_j(a_m)$ for all $i, j$ (where individuals in $i$ and $j$ are of course members of $D$); (3) An interpretation of the variables of $L$, $s$, is a function assigning a value in the domain to any variable of $L$. A pair $r^s$ is a valuation and a triplet $r^s_i$ is a world valuation. It is easy to see that such a model satisfies the Fregean Compositionality Principle, exemplified by a particular case (which can be generalized):

> The truth-value (extension) of an atomic sentence of the form "*a* is *P*" (*Pa*) is a function of the extensions of its parts: the name *a* and the predicate *P*. As a consequence, in order to ascertain whether Pa is true or false, one has first to identify the *denotatum* of the name *a*.[800]

As we have seen, the reasons for thinking that this sort of condition breaks down within quantum mechanics are not decisive. Dalla Chiara presents a counterexample in the case of a proper description, where the sentence 'there is exactly one *F*' is true yet the description term 'the $x$ which is *F*' fails to have a denotatum in the domain. To deal with this sort of case she proposes a "weak theory of descriptions". There are two strategies which she discusses. One involves weakening the constraints on the functions $r$ such that they need only be partial functions. The second involves modifying the requirement that the domain $D$ be a set, treating it instead as a *quaset*.

The first strategy works by requiring that, although there may be worlds where proper descriptions are denotationless, these worlds can be correlated

[799] The contrast here is with the standard characterizations of quantum logic which are Kripkean in the sense explicated here. Such logics, it is claimed, fail to accommodate absolutely indistinguishable particles (Dalla Chiara 1985, p. 190).

[800] Dalla Chiara 1985, p. 189.

to accessible worlds in which the description acquires a precise denotatum in *D*. This is extended to a treatment of proper names by the introduction of a *liberalized Kripke model*, which is a structure of the sort described above, except that *r* in this case is a partial function which associates extensional meanings to some but not all of the non-logical constants. Thus $r_i(a_m)$, if defined, is an element of *D* and $r(p_m^n)$, if defined, is an *n*-ary relation on *D*. Thus there will be worlds in which a name $a_m$ does not pick out a denotatum in *D*. However, proper descriptions of the sort 'The a which is *F*' are well behaved in the sense that a world in which a proper description is denotationless can be correlated with a world in which the term does denote (and denotes uniquely).

This approach retains an essentially classical ontology for which we still have a domain *D* of individual objects. We can think of this in two ways. It seems a natural way to think about Bohm's version of quantum mechanics. The fact that the functions $r_i$ are partial functions expresses epistemic ignorance about the exact behaviour of the particles, but the particles can be individuated and do have properties—their positions—which will serve to distinguish them one from another.[801] The other way of thinking about this is to consider the situation along the lines of Bohr's complementarity.[802] Here the domain *D* remains a set of classical objects and represents the classical resources we are limited to in our attempts to conceptualize the world. The partial functions then represent the restrictions complementarity places on our use of language if these classical concepts are not to clash with what is given in experience.

However, as mentioned above, Dalla Chiara has a second strategy for producing a semantics appropriate for quantum logics. Consider first the classical case of a system of *n* particles. We can name the elements and predicate properties of them, such as $Q_m^E$, which represents the claim that the value of some quantity $Q_m$ lies in *E*. If the domain of individuals at any time *t* is $D_t$ then $r_t(a_m)$ is some element of $D_t$ and $r_t(Q_m^E)$ is some subset of $D_t$. What of the situation for a system of *n* identical quantum particles in a state $f(t)$? Dalla Chiara poses two questions:[803] "(I) Does $f(t)$ determine a set of n elements in the standard set-theoretical sense? (II) Is the denotation-function definable in such a way that $r_t(a_n)$ univocally determines an element of $D_t$?"

She defines a liberalized Kripke model as before, where *I* is now the set of all unitary vectors in the Hilbert space $\mathcal{H}$, $\pm$ is the accessibility relation of

[801] See our discussion in Chapter 4, especially with regard to Belousek's work.

[802] In the neo-Kantian reading given by Folse 1985.

[803] Op cit., p. 199.

non-orthogonality, $P^U$ is the set of all sets $X$ of unitary vectors which are in one–one correspondence with subspaces of $\mathcal{H}$ and $D_f$ "represents a physical situation consisting of two physical objects, as is described by the vector $f$" (we quote Dalla Chiara's wording here because in view of what follows it seems perhaps question-begging). Possible worlds are associated with vectors in the Hilbert space, and, recalling our discussion above, it will be useful to have in mind a subset of these representing *physically* possible worlds.[804] A world $f$ verifies a sentence like $\mathcal{Q}_m^E a_1$ if the first subsystem certainly has a value for the quantity $\mathcal{Q}_m$ lying in E.[805]

What sort of admissible worlds do we have? We can have states like

$$\frac{1}{\sqrt{2}}(|u_1\rangle|d_2\rangle + |u_2\rangle|d_1\rangle)$$

but not states like $|u_1\rangle\ |d_2\rangle$. Dalla Chiara notes that states like $|u_1\rangle\ |d_2\rangle$ would represent worlds where the labels 1 and 2 had precise denotata (and proper descriptions such as "the electron with spin up" would also have precise denotata), but that such states are not admissible states for real systems. Instead she suggests that we treat these states as what she calls "Gedanken" states. We can then introduce a sense in which a "real" state (that is, one representing a physically possible world) verifies a statement of the type $\mathcal{Q}_m^E a_1$, or more generally "$a$ is $F$". A real state verifies "$a$ is $F$" if and only if *all* Gedanken states compatible with the original state can be correlated with a compatible state where the name $a$ is defined and where "$a$ is $F$" is true.

Consider our spin singlet state. At first sight it might seem that what we have is the following situation. We correlate the real state $\frac{1}{\sqrt{2}}(|u_1\rangle|d_2\rangle + |u_2\rangle|d_1\rangle)$ with a Gedanken state $|u_1\rangle\ |d_2\rangle$ which enables us to make sense of the notion that the real state verifies the sentence "electron 1 has spin up". But this will not do, for the singlet state is also compatible with a Gedanken state $|u_2\rangle\ |d_1\rangle$ which cannot be correlated to the state $|u_1\rangle\ |d_2\rangle$ in which it makes sense to talk of electron 1 having spin up.[806] Thus it seems that even in this weak sense there will be very few admissible states (that is, those representing physically

[804] Toraldo di Francia (1985) also finds this a useful distinction to make.

[805] Dalla Chiara notes that in general, $D_f$ will be a quaset, namely a collection of objects admitting of a cardinal but not an ordinal number (that is, in say a spin singlet state, we know there are two particles but it makes no sense to order them and say the first has spin up, the second spin down). We shall outline quaset theory below.

[806] In this sense, this requirement seems more stringent than that of 'mock individuality' where in order to give a name to an object all that is required is that we can treat the objects in such a way that we will find one in one state, the other in the other, although Toraldo di Francia notes that such names do not behave as rigid designators.

possible worlds) in which proper descriptions and singular terms in general will be meaningful, even in this weak sense.

At the conclusion of her account, Dalla Chiara maintains that the trend in 'traditional' semantics (citing Mill and Kripke) has been to connect the use of proper names with an ostensive function.[807] However, she urges, this function is not available in QM and hence names cannot be understood, in such situations, as mere *labels*. What the above considerations demonstrate is that "… one may regain a theory of *individual terms* (names and descriptions) which is not necessarily bounded to a *referential commitment*".[808] However, as we have seen, problems arise *within the account itself*. The source of these problems lies with the insistence on a descriptivist approach: if names are definite descriptions, then it remains opaque how we are to regard the labels in a superposition such as $\frac{1}{\sqrt{2}}(|u_1\rangle|d_2\rangle + |u_2\rangle|d_1\rangle)$ but not states like $|u_1\rangle\,|d_2\rangle$. Dalla Chiara's own attempt to spell this out by correlating this state with $|u_1\rangle\,|d_2\rangle$ understood as a 'Gendanken' state runs into the difficulty indicated above. This brings out a tension within the Dalla Chiara and Toraldo di Francia approach: quantum particles are regarded as 'anonymous' individuals and the issue, as we have noted, is how we can talk about such individuals. Their answer is two-fold: a liberalized Kripke semantics tied to a non-standard set theoretical description of the relevant elements of the domain. We have still to come to the latter, but the former is hamstrung by the dependence on a descriptivist view of names.

What are the alternatives? One would be to abandon the project altogether and follow Redhead and Teller in their insistence that the appropriate metaphysics for quantum particles is one of non-individuals. If one wishes to cleave to a descriptivist approach, this would appear to be the only option available. Thus we recall Maidens' claim that quantum particles cannot possess qualitative thisnesses, because of the entanglement. Hence, she concludes, they must be regarded as non-individuals.[809]

However, we have indicated above that one can still maintain some form of 'hybrid' approach to naming consistent with the metaphysics of quantum particles as individuals, particularly if one were to accept Gracia's distinction

[807] Dalla Chiara 1985.

[808] Ibid., p. 201.

[809] Maidens 1998, p. 85. Maidens actually prefers the term 'quasi-individuals', although we see no particular advantage to such terminology. Somewhat curiously, she also invokes Teller's non-supervenient relations in this context, where these are now regarded not as holding between individuals and thus veiling them, as it were, but between such quasi-individuals (Maidens op. cit., p. 85).

between the 'referring' and 'descriptive' functions associated with names. With states such as $|u_1\rangle\ |d_2\rangle$ these functions coincide, so that we are able to assert that $\frac{1}{\sqrt{2}}(|u_1\rangle|d_2\rangle + |u_2\rangle|d_1\rangle)$ s spin up'. In these cases, the description may serve to refer to the individual. However, in cases such as it may not. In these cases, we simply have to accept that the labels refer, but we cannot say *to which* individuals they refer, in the sense of giving an appropriate description. We can at least explain why we cannot give such a description (cf. Maidens' remark above regarding the Descriptivist approach): superpositions can be understood in terms of *non-supervenient relations* holding between the individuals, where these relations effectively hide or 'veil' the individuals and thus prevent the ascription of appropriate descriptions.

This may then enable us to respond to concerns about what the names could *mean* in the above situations. In particular, if the meaning of names is understood, at least in part, in terms of rigid designation, so that this meaning is cashed out, as it were, in terms of the stipulation of corresponding possible worlds, then it might seem as if we are caught in Kripke's dilemma above: we must then accept states such as $|u_1\rangle\ |d_2\rangle$ as corresponding to legitimate possibilities, but such states are not legitimate from the perspective of QM. The response we have delineated above is to effectively place constraints on this 'cashing out', so that not all such possibilities correspond to 'genuine', physical possibilities. But in that case, it might be insisted,[810] the names must be meaningless. When we are considering just the symmetric or anti-symmetric subspace, having discarded, as it were, the non-symmetric ones as not genuine possibilities,[811] the *symbols* $|u_1\rangle\ |d_2\rangle$ and $|u_2\rangle\ |d_1\rangle$ are to be understood as perfectly meaningless, as is the expression $[|u_1d_2\rangle + |u_2d_1\rangle]$, and, hence, also the labels themselves.

Alternatively, however, consider the parallel with well-formed formulae in propositional logic, say, where the string '*P*&' is taken to be meaningless, but neither the string '*P*&*Q*' nor the symbol '&' is.[812] Thus, in the context of the formalism of the symmetric and anti-symmetric subspaces, the strings $|u_1\rangle\ |d_2\rangle$ and $|u_2\rangle\ |d_1\rangle$ are understood as uninterpreted since they denote no states. However, it does not follow either that $|u_1d_2\rangle + |u_2d_1\rangle$ is meaningless or that the labels themselves are meaningless. In reply, the Compositionality Principle can be appealed to

[810] By Paul Teller, email communication.

[811] The context here is that of an email debate between Huggett and Teller over the latter's 'surplus structure' argument for the metaphysics of quanta, where Huggett rejects this argument.

[812] Huggett, email communication.

again:

> Please explain to me what '*P*&*Q*' means WITHOUT appealing to what the parts mean. Perhaps you will say that you can say what the ultimate parts mean in the QM case: '1' and '2' refer [to] individual particles. [$|u_1\rangle$ ] refers to a state in which particle [1] has spin up and similarly for [$|d_2\rangle$ ] etc. But you have acknowledged that [$|u_1d_2\rangle$ and $|u_2d_1\rangle$ ] are meaningless. So how are you going to explain what [$|u_1d_2\rangle + |u_2d_1\rangle$ ] means?[813]

Such an appeal undermines the parallel with propositional logic, since compositionality is obeyed with respect to such logic, of course, but may be viewed as problematic in QM, as we have indicated. If the principle is accepted, then explaining what '*P*&*Q*' means does indeed involve an appeal to what the parts mean: we explicate the truth of '*P*&*Q*' in terms of the truth of '*P*' and of '*Q*' and the meaning of the symbol '&'. On the quantum side, however, we must be careful in working our way back up from the components to the superposition, as we have already indicated. In the case of $|u_1\rangle$ , we can say that this string of symbols refers to a state in which particle 1 has spin up and *in this case*, the sense of reference is one in which the label both denotes the individual and 'stands for' the description in terms of which we may pick that individual out. The same can be said for the labels in both $|u_1d_2\rangle$ and $|u_2d_1\rangle$ , so in this sense, these latter expressions are not strictly meaningless.

Nevertheless, that does not imply that they must be accepted as genuine physical possibilities—this is precisely what the constraints on what can be stipulated as a possible world block. In other words, we can understand what these strings of symbols mean in a modal sense, just as we can understand the meaning of the strings of symbols representing the equations of classical mechanics. The problem lies with $|u_1d_2\rangle + |u_2d_1\rangle$ , and if one were to understand compositionality in the sense indicated by Dalla Chiara above, namely that in order to ascertain the meaning of 'Pa' (understood in terms of its truth or falsity), one has first to *identify* the denotatum of the name *a*, then such an expression would indeed appear to be meaningless, since we precisely have no means of making such an identification. However, if we reject this identification-first approach, then we can understand this expression as meaning that one electron has spin up and one has spin down, but we cannot say

[813] Teller, email communication. And, he continues, if such an explanation is given in terms of $|u_1d_2\rangle + |u_2d_1\rangle$ being a superposition of $|u_{>1}\rangle\ |d_2\rangle$ and $|u_2\rangle\ |d_1\rangle$ , then one is tacitly referring to the 'full' tensor product space which includes the subspaces supposedly ruled out as not genuine possibilities and hence one has re-inherited all the objectionable 'surplus structure'.

which because of the existence of these obscuring non-supervenient relations holding between the individuals concerned. In other words, the labels '1' and '2' have meaning in the sense of denoting individuals, but not in the sense of allowing us to distinguish which individuals have which properties, or satisfy which descriptions. It is only if one takes the sense of meaning to be entirely cashed out in the latter terms that one would be inclined to regard the above expression as meaningless.

Of course, what this suggests is the need for a non-Fregean semantics and, further, a non-standard set theory, since the above considerations raise obvious concerns regarding the principles of compositionality and extensionality. It is just such a theory that Dalla Chiara and Toraldo di Francia have attempted to elaborate in terms of quasets.

## 5.5 QUASETS

The central idea of quaset theory is that an assembly of quantum particles—electrons in an atom, say—has a cardinal but not an ordinal.[814] If we consider a helium atom, for example, we can conclude that there are two electrons in the assembly because the relevant state function depends on six coordinates and we can also ionize the atom and strip two electrons off, where we can distinguish them in the 'mock' sense seen above. However, within the assembly, the particles cannot be regarded as so ordered. Nevertheless, one can talk of subassemblies inside the assembly where each subassembly is determined by some property (corresponding to a quantum number); each subassembly possesses its own cardinality but, again, not an ordinality. Such assemblies and subassemblies are quasets and subquasets respectively.

The extension of the natural kind 'electron' cannot, therefore, be given by determining a set with a specified property and assuming the standard comprehension and extensionality principles. Instead, the extension of 'electron' can be defined as a quaset whose subquasets "… are all the electron systems that can—at least in principle—be detected experimentally".[815] The *intension* of the term 'electron', on the other hand, is given by the conjunction of the bundle of properties defining the nomological object.[816] Such properties are

[814] Toraldo di Francia 1985, p. 210; Dalla Chiara and Toraldo di Francia 1993, p. 268; Dalla Chiara, Giuntini and Krause 1998.

[815] Toraldo di Francia 1985, p. 211; Dalla Chiara and Toraldo di Francia 1993, p. 269.

[816] Ibid.

essential in that a particle which has all the properties of an electron except a mass of, say, $1.9 \times 10^{-25} g$ is not regarded as a peculiar electron, but as a different particle entirely (a muon in fact). Corresponding to a given intension, there may be more than one extension in the following sense:

> … once the intension of the term 'electron' has been stipulated, we have the possibility of recognizing—by theoretical or experimental means—whether a given physical system is an electron quaset or not; if yes, we can also enumerate all the quantum states available within it. But we can do so in a different number of ways! For example, take the spin. We can choose a z-axis and state how many electrons have $s_z = +1/2$ and how many have $s_z = -1/2$. But we could instead refer to the *x*-axis, or the *y*-axis, or any other direction, obtaining different sets of quantum states, all having the same cardinality.[817]

Although a particular name might have a well-determined and precise intension, in the case of quantal particles, the extension is 'strongly indetermined' in that the name lacks a precise reference. Thus, the absolute indistinguishability of such particles gives rise to 'uncertain and ambiguous denotation relations', as already noted. We shall now outline the formalism corresponding to this metaphysical package.

## 5.5.1 Quaset Theory[818]

Let us begin by recalling the obvious point that in various situations physicists are able to recognize, by theoretical or experimental means, whether a given physical system is an electron system or not. In these cases, they can also enumerate all the quantum states available within such a system, and they can do so in a number of different ways. For example, by considering the spin, one can choose the *x*-axis and state how many electrons have spin up and how many have spin down. However, we could instead refer to the *y*-axis or any other direction, obtaining *different collections* of quantum states, all having the same cardinality. This motivates the suggestion that micro-object systems present an irreducibly intensional aspect: generally they do not determine precise extensions and are not determined thereby. Accordingly, a basic feature of the quaset theory *QST* is to incorporate a strong violation of the extensionality principle.

[817] Dalla Chiara and Toraldo di Francia op. cit., p. 270.

[818] See Dalla Chiara and Toraldo di Francia 1993, 1995; Dalla Chiara, Giuntini and Krause 1998.

*QST* is formalized in classical first-order logic *with* identity (which is dealt with as a logical constant), and its language contains the following primitive non-logical concepts:

(i) one monadic predicate: *urobject* ($O$);
(ii) three binary predicates: the *positive membership* relation ($\in$), the *negative membership* relation ($\notin$), and the *inclusion* relation ($\sqsubseteq$). In the intuitive interpretation, $x \in y$ ($x \notin y$) is read as '$x$ certainly belongs to $y$' ('$x$ certainly does not belong to $y$') (see the explanation below);
(iii) a unary functional symbol: the *quasi-cardinal of* (*qcard*);
(iv) a binary functional symbol: the *quaset-theoretical intersection* ($\sqcap$).

Restricted quantifiers are assumed: $\forall_{Px}A$ stands for $\forall x(P(x) \rightarrow A)$, while $\exists_{Px}A$ stands for $\exists x(P(x) \wedge A)$. Further, the quantifier 'there exists exactly one' ($\exists!$) is understood as defined in the usual way.

We will present here only the minimal axiomatic nucleus of the theory. The notion of *quaset* is defined as follows:

**Definition 1** [**Quaset**] *A quaset is something that is not an urobject. In symbols,*

$$Q(x) := \neg O(x).$$

Then, the basic idea of a *quaset* is that of a collection of objects which has a well-defined cardinal, but in such a way that there is no way to tell (with certainty) which are the elements that belong to the quaset. The primitive predicates $\in$ (the positive membership) and $\notin$ (the negative membership, which is not the negation of the former) help in expressing this. The postulates, to be stated below, imply that $z \in y$ entails $\neg z \notin y$, but not the converse. So, it may be the case that it is false that $z$ certainly does not belong to $y$, but this does not entail that $z$ (certainly) belongs to $y$. The elements $z$ for which it may be said that 'it is false that they certainly do not belong to $y$' might act as members *in potentia* of $y$.[819] Since the cardinal of the quaset is well defined, there is a kind of 'epistemic' indeterminacy with respect to the elements of a quaset. The postulates below provide the grounds for the whole theory:

**Ax1** All that has elements is a quaset: $\forall x \forall y\ (y \in x \rightarrow Q(x))$.

**Ax2** The inclusion relation $\sqsubseteq$ between quasets is a partial order (reflexive, anti-symmetric and transitive).

[819] Dalla Chiara and Toraldo di Francia themselves do not use this terminology.

In the intended interpretation, $\subseteq$ has an intensional meaning: $x \subseteq y$ can be read as 'the concept *xinvolves* (or *implies*) the concept *y*'.[820]

**Ax3** Suppose that something certainly does not belong to a given quaset. Then it is not the case that it certainly belongs to our quaset, but generally not the other way around: $\forall x \not\forall y (x \notin y \rightarrow \neg x \in y)$.

As a consequence, a strong *tertium non datur* principle ($x \in y \vee x \notin y$) fails and indetermined membership relations are possible (in accordance with the quantum uncertainty relations).

**Ax4** Intensional inclusion implies extensional inclusion (but not the other way around):

$$\forall x \forall y (x \subseteq y \rightarrow \forall z (z \in x \rightarrow z \in y) \wedge (z \notin y \rightarrow z \notin x))$$

**Ax5** Any quaset has exactly one quasi-extension, where the *quasi-extension* of a quaset *x* is the unique quaset that certainly contains all the elements of *x* and certainly does not contain all the other entities:

$$\forall_Q \exists!_Q y \forall z ((z \in y \leftrightarrow z \in x) \wedge (z \notin y \leftrightarrow \neg z \in x))$$

Axiom 5 justifies the definition of a unary functional symbol *ext* (the quasi-extension of *x*). (Definition 3.2.)

**Definition 2** (**The quasi-extension of a quaset**)

$$\forall x \forall y (y = ext(x) \leftrightarrow \forall z (z \in y \leftrightarrow z \in x) \wedge (z \notin y \leftrightarrow \neg z \in x))$$

**Definition 3***Sets are quasets that are identical with their quasi-extension.*

One can easily show that sets satisfy the Extensionality Principle. The extension of an empty quaset (which turns out to be trivially a set) is postulated.

**Ax6** [The empty quaset] There exists a quaset that necessarily does not contain any element: $\exists_{Q y \forall x} (x \notin y)$.

*QST* contains a copy of ZF (Zermelo-Frankel set theory), obtained by restricting the universe of quasets to *sets* only. For any formula *A* of ZF, let $A^x$ be the corresponding formula of *QST* relativized to sets. Then,

**Ax7** If *A* is any instance for an axiom of *ZF*, then $A^x$ is an axiom of *QST*.

[820] Dalla Chiara 1987.

The notion of quasi-cardinality of a quaset is introduced as follows:

**Ax8** Any quaset has a unique quasi-cardinal, which is a cardinal number:

$$\forall x \forall y (x \subseteq y \rightarrow qcard(x) \leq qcard(y))$$

A careful remark is in order here. The axiom says that the quasi-cardinal is a cardinal and thus, if the concept of cardinal is defined in the usual way, that is, as a particular ordinal, then a quasi-cardinal *is* an ordinal and hence it may appear that there is no sense in saying that a quaset may have a cardinal but not an ordinal, as we have suggested above. This should be understood in the sense that the concept of cardinal (or of counting) should be taken differently from standard mathematics (Zermelo-Fraenkel set theory), perhaps in the sense of Frege-Russell. This is of course interesting, but we shall not pursue the point here. It will be recalled in the context of quasi-sets in Chapter 7.

**Ax9** Quasi-cardinals and cardinals coincide for sets:

$$\forall x (Z(x) \rightarrow qcard(x) = card(x))$$

**Ax10** The quasi-cardinal of a subquaset is less or equal than the quasi-cardinal of the whole quaset:

$$\forall x \forall y (x \subseteq y \rightarrow qcard(x) \leq qcard(y))$$

**Ax11** The quasi-cardinal of a quaset is greater or equal than the quasi-cardinal of its quasi-extension:

$$\forall x (qcard(x) \geq qcard(ext(x)))$$

**Ax12** $\sqcap$ represents a weak conjunction for quasets. This weak conjunction coincides with the usual set-theoretical intersection in the case of sets:

$$\forall x \forall y ((x \sqcap y \subseteq x \wedge x \sqcap y \subseteq y) \wedge (Z(x) \wedge Z(y) \rightarrow x \sqcap y = x \cap y))$$

As a consequence, a separation procedure may be applied. Notice that our axioms do not require that proper quasets (that are not sets) exist. From an intuitive point of view, the quasi-extension of a proper quaset does not represent an adequate semantic counterpart for the usual notion of extension. Think for instance of the fact that the quasi-extension of a quaset, whose quasi-cardinal is greater than 0, might have an empty quasi-extension.

What can be said about the validity of Leibniz's Principle in this theory? Are non-identical objects always distinguished by a property, represented by a quaset? As expected, the answer is negative. Namely, the implication

$$\neg x = y \rightarrow \exists_Q z (x \in z \wedge y \notin z)$$

generally fails, and so the classical set-theoretical argument founded on the theorem of ZF:

$$\neg x = y \rightarrow x \in \{x\} \wedge y \notin \{x\}$$

obviously cannot be repeated here, for nothing guarantees that the singleton of $x$ (which should be the characteristic property of $x$) exists and that $x$ certainly belongs to it.

## 5.6 CONCLUSION

What we have tried to do in this chapter is to develop the particles-as-individuals package a little further by considering the treatment given to the particle labels in the quantum context. As we have indicated, there are ways in which such labels can be regarded as names and arguments to the effect that this is problematic are themselves contentious. Gracia's account, in particular, appears to offer an appropriate metaphysical framework in which the particles can be considered as named individuals for which distinguishing descriptions cannot be given. We have concluded by suggesting that Dalla Chiara and Toraldo di Francia's quaset theory provides the appropriate formal counterpart.[821] Understood in this way, it would not be the most appropriate formal framework for capturing the notion of 'non-individuality' as expressed in the Received View. One way of doing that would be to represent this notion by abandoning the condition of self-identity for quantum particles and constructing a formal system on that basis. We shall now consider just such a construction.

[821] This does not necessarily correspond to Dalla Chiara and Toraldo di Francia's own understanding of their work. As we have seen, there exists a certain tension in this understanding.

# 6
# A Problem for Present-Day Mathematics

> The development of the foundations of physics in the twentieth century has taught us a serious lesson. Creating and understanding these foundations turned out to have very little to do with the epistemological abstractions which were of such importance to the twentieth century critics of the foundations of mathematics: finiteness, consistency, constructibility, and, in general, the Cartesian notion of intuitive clarity. Instead, completely unforeseen principles moved into the spotlight: complementarity, and the nonclassical, probabilistic truth function. The electron is infinite, capricious, and free, and does not at all share our love for algorithms.
>
> Manin 1977, pp. 82–3

During the International Congress of Mathematicians, held in Paris in 1900, the great mathematician David Hilbert presented a list of 23 Problems of Mathematics which in his opinion should occupy the efforts of mathematicians in the century to come. To solve one of the problems became a way of achieving something significant in mathematics, and several Fields medals were awarded for this kind of endeavour.[822] The sixth problem of his celebrated list dealt with the axiomatization of the theories of physics; it was proposed "to treat in the same manner [as Hilbert himself had done with geometry], by means of axioms, those physical sciences in which mathematics plays an important part".[823] In the twentieth century, much was done along these lines, in continuation of those efforts already developed in the previous period, as remarked by Hilbert himself in his aforementioned paper.[824] In 1974, the American Mathematical Society sponsored a conference to evaluate and

[822] See Hilbert 1902. The Fields Medal is the highest scientific award for mathematicians, presented every four years at the *International Congress of Mathematicians*. It is the closest analogue to a Nobel Prize, which is not awarded for mathematics.

[823] Hilbert, op. cit.

[824] For a general account on the problems, see Corry 1997, Wightman 1976.

to explore the consequences of Hilbert Problems.[825] One of the interesting implications of this meeting was that a new list of Problems of Present Day Mathematics was proposed. According to Felix Browder, the editor of the Proceedings, this list was initiated by Jean Dieudonne through correspondence with mathematicians throughout the world, and was further lengthened with other problems collected by Browder himself.[826]

## 6.1 THE STATEMENT OF THE PROBLEM

The first problem of this new list deals with the foundations of mathematics, and was stated by the mathematician Yuri I. Manin; in item (b) (cf. quotation below), he makes reference to the need for questioning the paradigm of classical set theory ('Cantor's paradise' referred to in the statement of the problem) on the basis of the treatment of collections of indistinguishable elementary particles in quantum mechanics, which (as he suggests) cannot be considered as standard 'sets'. It was then suggested that a 'new language' should be developed for such a purpose. Let us consider how Manin states his problem:

> In accordance with Hilbert's prophecy, we are living in Cantor's paradise.[827] So we are bound to be tempted.
>
> Most mathematicians nowadays do not see any point in banning infinity, nonconstructivity, etc. Gödel made clear that it takes an infinity of new ideas to understand all about integers only. Hence we need a creative approach to creative thinking, not just a critical one. Two lines of research are naturally suggested.

(a) To find out new axioms of (more or less naive) set theory, demonstrably efficient in number theory. Most advanced new methods (*l*-adic cohomology) should be explored thoroughly. Are they readily formalized in Zermelo-Fraenkel or Gödel-Bernays systems? Can we use in necessary categorical constructions only known axioms, or has something new already slipped in?

(b) We should consider the possibilities of developing a totally new language to speak about infinity.[828] Classical critics of Cantor (Brouwer et al.) argued that, say, the general choice axiom is an illicit extrapolation of the finite case.

[825] See Browder 1976.

[826] See 'Problems of Present Day Mathematics', organized by Browder, op. cit., pp. 35–79.

[827] 'Hilbert's prophecy' refers to Hilbert's well-known dictum that "No one shall drive us out of the paradise which Cantor has created for us" (Hilbert 1964; the 'paradise' is set theory and the phrase was coined in the context of the criticism which set theory suffered in the beginnings of the twentieth century.

[828] Set theory is also known as 'the theory of the infinite'.

> I would like to point out that this is rather an extrapolation of commonplace physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behavior. Even 'sets' of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the 'set' of grains of sand. In general, a highly probabilistic 'physical infinity' looks considerably more complicated and interesting than a plain infinity of 'things'.
>
> Certainly there are no a priori reasons to choose fundamental concepts of mathematics so as to make them parallel to those of physics. Nevertheless it happened constantly and proved extremely fruitful.
>
> The twentieth century return to Middle Age scholastics taught us a lot about formalisms. Probably it is time to look outside again. Meaning is what really matters.[829]

It seems natural to ask why Manin would suggest that the 'standard' axioms for sets are inadequate for representing collections of indistinguishable objects. In order to analyse such a claim, we need first to consider the basis of 'standard' set theories (as we shall refer to them from now on),[830] mainly in order to show that they involve a theory of identity which takes the elements of a set (even the *Urelemente*, if they are admitted by the theory) to be *individuals* of a kind. In short, this 'theory of identity' contrasts with the Received View of quantum entities as absolutely or 'strongly' indistinguishable entities, as we have discussed, and cannot provide the grounds for treating 'truly' indistinguishable non-individuals.

The need for such a 'new language' of 'sets' can perhaps be reinforced by Manin's own statement that "quantum mechanics does not really have its own language [but] uses a certain fragment of the language of functional analysis".[831] Manin's suggestion seems to constitute a different way of saying that quantum theory demands a different kind of logic.[832] In this chapter, we shall try to make sense of this suggestion and, in doing so, to motivate the introduction of a mathematical formalism (termed Quasi-Set Theory) which we think can be useful in responding to Manin's Problem. This theory will be presented in the

[829] Manin 1976.

[830] As will become clear below, this refers to practically all known 'classical' theories of sets, like Zermelo-Fraenkel, von Neumann-Bernays-Gödel or Kelley-Morse or whatever theory encompassing the traditional theory of identity. We shall leave the details to be explicated later.

[831] Manin 1977, p. 84. The word 'sets' was put into quotation marks above just to emphasize the point that collections of indistinguishable objects cannot be viewed as *sets* as described by standard set theories, that is, as collections of distinguishable elements. This is one of our central claims in this part of the book.

[832] Here we shall consider as 'logic' all of a theory's underlying formal apparatus, which may (mainly with regard to physical theories) encompass more than first-order logic.

next chapter. It is important to realize that quasi-set theory may be said to be *inspired* by the idea that the concept of identity might not be applicable to elementary particles, as Schrödinger claimed. The limitation imposed on the concept of identity will offer us the opportunity to elaborate a mathematical theory in which we can talk of indistinguishable but not identical objects, as we will see. Of course, the question arises whether, in applying such a theory to physics, it is necessary to fully develop it; perhaps for the purposes of physics, we don't need, for instance, *all* the mathematics that can be developed within such a theory, but only part of it. However, we shall not pursue that particular discussion here.[833] We shall limit ourselves to describing the main traits of the theory, and to presenting some examples of its possible uses in the quantum domain. Before we get to that point, let us motivate our approach.

As is well known, ever since the pioneering work of von Neumann in the 1930s, the claim that quantum theory demands a new kind of logic has been the topic of numerous discussions on the foundations of the theory. Typically, contributors to these discussions have questioned classical (propositional) logic, generally by accepting that the distributive laws, like $\alpha \vee (\beta \wedge \gamma) \leftrightarrow (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$, do not hold in the quantum domain.[834] Although we may find in the literature certain elements of mathematics motivated by aspects of quantum mechanics (see below), the field of quantum logics has not been greatly developed by systematic discussions in which doubt is cast on the 'more sophisticated' features of mathematics, namely, those which serve to express concepts like derivatives, differential equations and the like; in other words, set theory. These forms of logic have been developed independently of the kinds of foundational problem considered here, becoming (roughly speaking) something like the algebraic study of the lattice of all closed subspaces of a Hilbert space, and are in most cases, limited to the propositional level.[835] In a certain sense, quantum logic became a part of algebra, effectively

[833] See, for example, Scheibe 2001, VIII.38.

[834] It is important to note that in such a domain, or better, in the so-called 'quantum logics', the logical operators are not the classical ones, and that it is this which causes the 'failure' of the distributive law. In other words, the distributive law does not hold for these 'quantum operators'. Having said that, we shall continue to follow the traditional line of saying that the distributive law is 'violated' in the quantum domain.

[835] These systems have, of course, been developed as 'true logics', to use Putnam's phrase (Putnam 1974). See also Dalla Chiara 1986 or Mittelstaedt 1978. Historical details may be found in Chapter 8 of Jammer 1974. Recently, Dalla Chiara and Roberto Giuntini have given an extensive and detailed presentation on quantum logics in xxx.lnal.gov/list/quant-ph/0101, forthcoming in the *Handbook of Philosophical Logic*. See also Dalla Chiara, Giuntini and Greechie, R., 2004.

keeping its distance from the concerns involved in the foundations of quantum theory.[836]

Nevertheless, it must be admitted that the study of quantum logics has led to the development of some interesting mathematical theories, in particular through the introduction of 'quantum sets' of different kinds. For instance, it is well known that the so-called Boolean valued model technique enables us to construct a model for set theory (say, Zermelo-Fraenkel set theory) starting from a complete Boolean algebra.[837] Thus G. Takeuti asked: what kind of universe can we obtain if we apply such a technique not to a Boolean algebra, which is 'the logic of classical mechanics', but to an orthomodular lattice, taken as 'the logic of quantum mechanics'?[838] In other words, instead of taking truth-values in a complete Boolean algebra, Takeuti took them from a complete orthomodular lattice. The resulting theory is his 'quantum set theory', which is still to be further developed, and its relationships with quantum physics adequately explored (perhaps due to the fact, already remarked by Takeuti himself, that quantum mathematics is "extremely difficult … [although it has] a very rich mathematical content").[839]

Other kinds of 'set' theories arising from quantum logic can also be found in the literature. For instance, Bell proposed a way of distinguishing classical logic from quantum logic by developing a set theoretical way of speaking of superpositions and the incompatibility of attributes, which are concepts taken as characteristic features of quantum theory.[840] His resulting concepts of *proximity space* and of *quantum sets*, have also not been much pursued in the literature, their theoretical importance notwithstanding.[841] Other related mathematical theories have originated from the same source, such as Nishimura's 'empirical set theory',[842] and Finkelstein's quantum set theory, just to mention two.[843]

Although we are not trying to provide a complete account of the developments proposed in recent years, the above references suggest that there has been at least some tendency to question the set theoretical apparatus related to quantum theory, and not only the underlying propositional logic. However, none of these theories deal specifically with collections of strongly

[836] See, for example, Hooker 1973; Jammer 1974, p. 361.

[837] Bell 1977. Boolean models can also be found in Manin's book (1977).

[838] For details concerning these 'logics', see Jauch 1968.

[839] See Takeuti 1981, pp. 303–22; Dalla Chiara and Giuntini op. cit., Dalla Chiara, Giuntini and Greechie, op. cit.. See also Dunn 1981.

[840] Bell 1986.

[841] Although certain developments in this direction were presented in Resconi et al. 1999.

[842] Nishimura 1993, 1994, 1995.

[843] Finkelstein 1982.

indistinguishable objects, as required by Manin, and in particular it is not clear how they consider individuation.[844]

Before we present our own account, let us put it in a more general context by making a remark on the plurality of formal systems which can be constructed on the basis of philosophical speculations in relation to Hilbert's sixth problem. An empirical domain, let us call it *D*, is in general approached by what we usually (and imprecisely) call a *theory* (perhaps it would be better to say that we have only a 'pre-theory'), which is only informally stated, with no well-defined boundaries and clear principles, like a defined language, logic and other details; at this stage, such a collection of concepts and rules cannot be identified with a set of sentences closed by deduction, as logicians call a 'theory'. Since it is not rigorously stated, this pre-theory is only ideally closed, in the sense that the consequences of its (not clearly stated) basic rules still belong to it, given that concepts like logical consequence are not yet made precise. In general the pre-theory's underlying logic may not be fully articulated (typically, classical logic has been assumed, but only informally). Furthermore, in general scientists do not use only deduction in approaching *D*, but also inductive ways of reasoning of several forms, such as non-monotonic and defeasible forms of reasoning. Frequently they may add new concepts to the old framework, so that problems like the consistency between these new concepts and the old ones are in general not regarded as something which deserves further investigation (a well-known example of an inconsistent theory would be Bohr's theory of the atom).[845] Perhaps an empirical domain *D* may admit several non-equivalent axiomatic approaches (perhaps involving also some forms of 'inductive logics'), maybe based on different kinds of logic, using different languages and principles and, depending on certain criteria, one of them can be chosen for some particular purpose, say to emphasize a certain aspect of *D* that the scientist wishes to delineate.[846]

[844] A new way of looking at this issue was recently proposed by Tzouvaras 2004.

[845] da Costa and French 2003, Chapter 5. Another and perhaps more interesting example can be taken from present-day physics; as Arthur Jafee says, "[y]et in spite of these great successes, we do not know if the equations of relativistic quantum electrodynamics make mathematical sense. In fact the situation poses a dilemma. As a consequence of the renormalization group theory, most physicists today believe that the equations of quantum electrodynamics in their simple form are inconsistent; in other words, we believe that the equations of electrodynamics have no solutions at all!! (Jafee 1999). For a more sanguine view of renormalization, see Teller 1995.

[846] All these possible 'theories' about *D* can (in principle) be joined by a more general 'meta-theory' whose underlying logic would be 'paraclassical'. Alternatively, perhaps we could use a form of Jaśkowski's logic. On paraclassical logic, see da Costa and Krause 2004. On Jaśkowski's logic, see da Costa and Dubikajtis 1977; da Costa, Krause and Bueno 2005. Roughly speaking, with these logics it does not follow in general that from two contradictory propositions anything can be derived, as in classical logic—and in most systems in general. Thus, the different axiomatic approaches do not cause the whole system regarding *D* to be trivial.

With regard to the quantum domain, it can be approached by several distinct theories, such as those based on the Hilbert-space formalism, the algebraic approach, or even by using Bourbaki's species of structures, as developed by Ludwig.[847] We cannot regard, in principle, all these possible approaches as equivalent to one another, since the notion of 'equivalent' theories—in the sense we are concerned with here—can be formally defined only in connection with well-defined mathematical contexts, generally by using model-theoretical techniques.

At this point, the realism-antirealism debate intrudes. The anti-realist would typically insist that the search for the 'right' approach to *D* may be senseless, and would suggest that the choice of one of these partial approaches as the 'correct' one is based on criteria which are ultimately, and simply, pragmatic. The realist will want to maintain that at least some of these approaches are equivalent, at least in terms of theoretical (as well as empirical) content and of those remaining, a choice can in principle be made for sound, epistemic reasons. However, as we have emphasized here, if that theoretical content is taken to have a metaphysical component, in the sense that the realist's commitment to a particular ontology needs to be articulated in metaphysical terms, and in particular with regard to the individuality or non-individuality of the particles, then the realist appears to face a situation in which there are two, metaphysically inequivalent, approaches between which no choice can be made based on the physics itself. And, of course, our claim here—to be articulated in the rest of the book—is that these different approaches correspond to different logico-mathematical axiomatizations in terms of standard and non-standard set theories. The choice for the realist is stark: either fall into some form of antirealism or drop the aforementioned metaphysical component and adopt an ontologically less problematic position.

Thus, as we shall make clear below, although there are different 'theories' of the quantum domain, all the forms of 'quantum mathematics' considered until recently, in so far as they have been constructed with the resources of standard set theories, remain compromised with a theory of identity which leads to

[847] Ludwig 1990, Scheibe 2001. Consider the following example: suppose the informal domain is microphysics described by quantum theory as standardly understood in terms of the Copenhagen Interpretation, encompassing complementarity. Then we can consider its 'foundational' logical basis as P. Février's tri-valued logic (Février 1951) or even a form of 'paraclassical logic'; see da Costa and Krause 2004 for a discussion of both aspects.

philosophical problems when it comes to the form of indistinguishability we have been exploring here. Within these mathematical frameworks, the usual way of dealing with indistinguishability is by the introduction of some 'additional' symmetry postulates, such as the Symmetrization Postulate, as discussed in Chapter 4. Hence, if our aim is to consider *quanta* as legitimate non-individual entities, taken as such *right at the start*, as suggested by Post,[848] a different logic-mathematical (set-theoretical) framework is in order. We shall return to this point in more detail below, but let us emphasize that this is a legitimate philosophical question. The standard argument that 'there are other alternatives', developed within standard logic and mathematics, to capture the indistinguishability of quanta, is precisely what we are trying to avoid here. Our main goal is to pursue Post's suggestion, which is obviously linked to the Manin problem mentioned above, and to follow Schrödinger's intuitions regarding the lack of sense in applying the concept of identity (as given by the 'classical' theory) to these entities. It is our view that the quantum realm still lacks an adequate mathematical framework which enables us to deal with entities which are treated as non-individuals *right at the start* for, as we shall argue below, standard set theories do not provide such tools, and the standard ways of dealing with this concept within the usual set theoretical frameworks only mask the basic philosophical problem. Our claim is that quasi-set theory may be such a theory, and although at the present stage it still does not provide a complete underpinning of quantum mechanics, it nevertheless comes close to the kind of mathematical description of systems of indistinguishable objects that we want.

## 6.2 THE USE OF 'STANDARD LANGUAGES'

Let us return to the specific case of quantum theory in order to support our claim that standard set theories do not provide adequate mathematical tools for dealing with collections of 'legitimate' indistinguishable entities, although they can be used for dealing with such entities if we 'forget' that they are *individuals* of a sort (according to the underlying mathematics) and treat them as indistinguishable in some way. This point will be explicated in what follows.

[848] Post 1963.

We begin by recalling that in his 'What is an elementary particle?',[849] Schrödinger claims that physics begins with everyday experience, and, in particular, the macroscopic bodies of our surroundings. He writes that

> [w]e have taken over from previous theory [classical mechanics] the idea of a particle and all technical language concerning it. This idea is inadequate. It constantly drives our mind to ask for information which has obviously no significance. Its imaginative structure exhibits features which are alien to the real particle. (…) The particle (…) is not an identifiable individual. (…) *It is not at all easy to realize this lack of individuality and to find words for it.*[850]

It is clear that Schrödinger is emphasizing the limitations of the language of classical physics for dealing with quantum entities. According to him, the probabilistic interpretation seems to be vague as to whether it refers to one particle or an ensemble of them. In other words, it would not be clear whether it indicates the probability of finding 'the' particle or 'a' particle or even an average number of them in a certain small volume.[851] He then suggests that the way to overcome this difficulty would be by using the second quantization approach, which doesn't deal with 'individual' particles at all, retaining simply that which expresses only the number of such particles in each state. The important point to be emphasized here is that, according to Schrödinger, the use of standard languages (of classical physics) "drives our mind to ask for information which has obviously no significance. Its [the language's] imaginative structure exhibits features which are alien to the real particle."[852]

One might say that it is not only classical mechanics that uses concepts taken from our ordinary experience, but classical mathematics (and logic) in a certain sense, do so as well.[853] For instance, when someone thinks of a *set*, he or she intuitively thinks of a collection of 'classical objects', like those of our surroundings, hence *individuals*, as we have said in the Introduction. So, based as they are on standard mathematics, physical theories become in a certain sense dependent (in a sense to be explained below) on such mathematics, and a physical theory based on classical logic and mathematics cannot dismiss the theorems of such a logical and mathematical basis.

[849] Schrödinger 1998; also in Schrödinger 1957, pp. 193–223; our references are from this last version.

[850] op. cit., pp. 204–5, our emphasis.

[851] op. cit., pp. 205–6.

[852] Schrödinger 1957. See also Bitbol 1992.

[853] As suggested by da Costa, "[t]he notions of object, of property and of relation of both Aristotelian logic and of present day mathematical logic, came from the static and Euclidian view of reality" (da Costa 1980, p. 120).

Furthermore, from the perspective of the semantic approach to theories, as Suppes once remarked, "there is no theoretical way of drawing a sharp distinction between a piece of pure mathematics and a piece of theoretical science (…). From the philosophical standpoint there is no distinction between pure and applied mathematics, in spite of much talk to the contrary".[854] Since we can regard even quantum mechanics as something which can be theoretically 'reduced' to set theory, in the sense that whatever concepts it uses can be defined in set theoretical terms[855] then the set-theoretical definitions of a quantum theory (and the same holds for other theories, like thermodynamics for example, as noted by Suppes)[856] are on all fours with the definitions of the purely mathematical theories of groups, rings, etc. Thus, since we are trying to pursue the idea of considering quanta as non-individuals, we need first to look at how classical mathematics and logic deal with individuality and identity.

## 6.2.1 The Lack of Identity

Let us begin by recalling Schrödinger's insistence that our usual ways of considering 'objects' (we may say, 'individuals') should be abandoned: "[w]hen a familiar object re-enters our ken, it is usually recognized as a continuation of previous appearances, as being the same thing. The relative permanence of individual pieces of matter is the most momentous feature of both every day life and scientific explanation".[857] Thus, he continues, we cannot ensure that an observed object which disappears momentarily from our eyes is the same one we have observed before, so that we may have doubts in asserting that it is the same as that which has disappeared: "[y]ou may not be able to decide the issue, but you will have no doubt that the doubtful sameness has an indisputable meaning—that there is an unambiguous answer to your query".[858] However, he continues, "[i]n the new turn of atomism that began with the papers of Heisenberg and of de Broglie in 1925, such an attitude has to be abandoned".[859] This 'new atomism' seems to suggest that we are faced

[854] Suppes 1992; see also his 2002, pp. 20 ff.

[855] The Hilbert-space formalism exemplifies this point, since it is completely developed in a set-theoretical framework; let us remark that we are not questioning the completeness—in Einstein's sense—of the theory here.

[856] op. cit.

[857] Schrödinger 1957, p. 209.

[858] Ibid., pp. 209–10. As we have already noted, he links this tendency to the notion of individuality taken from the ancient Greek atomists and which continues up to Dalton and beyond.

[859] Ibid.

with a new kind of object, an object to which even fundamental concepts like identity have no meaning.[860]

In discussing the case where one is tempted to say that one and *the same* elementary particle was observed at different instants of time, Schrödinger writes that

> [t]he circumstances may be such that they render it highly convenient and desirable to express oneself so, but it is only an abbreviation of speech; for there are other cases where the 'sameness' becomes entirely meaningless; and there is no sharp boundary, no clear-cut distinction between them, there is a gradual transition over intermediate cases. And I beg to emphasize this and I beg you to believe it: It is not a question of our being able to ascertain the identity in some instances and not being able to do so in others. *It is beyond doubt that the question of 'sameness', of identity, really and truly has no meaning.* (Our emphasis.)[861]

As we have already emphasized, Schrödinger is saying more than that there is simply a difficulty in recognizing an object once observed as being the same as the one we are observing now. The problem is not epistemological, but 'ontological'; furthermore, it does not simply concern *genidentity*, or identification through time, but *identity* itself.

Suppose that we accept for a moment what we have called the 'Received View', according to which the quanta are entities devoid of identity. If Schrödinger is right, then Quine's well-known ontological criterion (see Chapter 4) according to which there is no entity without identity, comes into question, for in such a Quinean ontology there would be no indistinguishable quanta. Even without trying to decide this issue, we may adopt the foundational point of view and consider how we can express this supposition in mathematical terms. In other words, we shall try to make sense of the 'lack of identity' of certain entities in such terms. The basic question is: how can we treat them collectively if they cannot be regarded as 'individuals'? Here of course we are supposing a strong relationship between individuality and identity. This is the route we have taken in this book, for we have characterized 'non-individuals' as those entities for which the relation of self-identity $a = a$ does not make sense (cf. the Introduction).

The language of standard mathematics (that is, extensional set theory and, without loss of generality, let us assume the Axiom of Regularity), as we shall see below, is based on the assumption that a *set* is a collection of *distinguishable*

[860] Bachelard 1940. These ideas can also be taken to apply to more recent 'quantum objects', like strings and branes.

[861] Schrödinger 1952, pp. 17–18.

objects. The apparently innocuous question above has important philosophical implications in considering a possible language for quantum physics which should express indistinguishability right at the start, as we shall now indicate.

To begin with, let us insist on the importance of a mathematical language in the presentation of a scientific theory. If we follow a reductionist programme, then it should suffice to refer to set theory, although in principle we could base (at least part of) our physics on another framework, such as category theory, or on some higher order logics or even on some kind of mereology, to mention some of the alternatives. Despite the differences among these mathematical theories, in what follows we shall talk in general terms and restrict our discussion to sets.

In order to explain our central point, we might suppose that the axioms of an axiomatic theory *T* can be divided into three parts: (1) the 'logical axioms' (say, the axioms for the classical first-order predicate calculus with identity), (2) the 'mathematical axioms' (say, the Zermelo-Fraenkel set theory) and (3) the 'specific axioms', which depend on the particular theory being considered. Of course this distinction is only schematic. Generally, as happens with mathematical theories, only the third group of axioms is mentioned, the other two being assumed implicitly. So, when the mathematician mentions axioms for groups, real numbers, Euclidean geometry, topological spaces and so on, he or she does not pay attention to the 'most basic axioms' of levels (1) and (2), which are usually implicitly understood as those of classical logic and set theory, notwithstanding that they do form part of the axiomatics, as shown for instance by Bourbaki.[862] The same of course holds for the axioms of physical theories, including quantum mechanics, for in assuming things like Borel sets, tensor products, orthonormal vectors, probability measures and the like, all of standard mathematics (read: set theory and classical logic) is being supposed.[863] Furthermore, the deductions one makes from the axioms follow the rules of classical logic; so, our above scheme comprises the central assumptions usually made when rendering physics 'rigorous'. In addition to all of this, the study, say, of the models of physical theories should take into account the fact that these models are models also of their logical and mathematical axioms, and not only of the 'specific' axioms. So, at least ideally, the models of quantum theory should capture what the theory determines;

[862] Bourbaki 1968.

[863] Mackey 1963. Mackey's axioms are also mentioned in Jammer 1974, pp. 385 ff.

in particular, these models should also give an account of non-individuality. And, as we have already said, an empirical domain of knowledge might be captured by more than one form of axiomatisation and in particular, focusing for the moment on 'level 2', by more than one axiomatic mathematical framework.

Let us now have a look at how classical logic and mathematics deal with the concepts of individuation and identity in order to pin down what seems to be wrong in the context of responding to the Manin Problem.

## 6.3 IDENTITY IN CLASSICAL LOGIC AND MATHEMATICS

In the next sections, we shall use the symbol ≈ for identity; the motives for this choice will be made clear below.

As we have said, classical languages (that is, the languages of classical logic and set theory) talk essentially about *individuals* and collections or properties of them (or of other higher-order collections or properties etc., depending on the level of the language considered). Thus, the intuitive idea of having in the language something (in particular) to express the identity of individuals is to have a way of expressing whether 'two' individuals of the domain are the same entity or not. The standard understanding of how this can be achieved is linked to a tradition which goes back to Leibniz. Leibniz's aphorism *Eadem sunt quorum unum potest substitui alteri salva veritate* has been taken as the motto for the appropriate way of characterizing identity in mathematics. Frege took possession of Leibniz's dictum for his 'definition' of equality in his *Die Grundlagen der Arithmetik*[864] but, as Church remarked, in such a phrase there is a confusion between use and mention, for "*things* are identical if the *name* of one can be substituted for that of the other without loss of truth".[865]

This confusion was corrected nine years later in Vol. 1 of Frege's *Grundgesetze der Arithmetik*,[866] but, there, instead of a definition, what we find is an axiom, written by Church as $\varphi(x \approx y) \supset \varphi((F)[F(x) \supset F(y)])$ (except that he used = instead of our ≈). Previously, Frege used a Hilbert style (as it was later called) axiomatics in his *Begriffsschrift* from 1879,[867] namely, axioms which

[864] Frege 1884/1969.

[865] Church 1956, p. 300, fn. 502.

[866] Frege 1893.

[867] Frege 1879.

correspond to the following (for any *a* and *b*):

$$(F1)\, a \approx a$$
$$(F2)\, a \approx b \longrightarrow (F(a) \longrightarrow F(b)) .$$

It should be realized that at that time there was no clear distinction between first and higher-order languages, something which was explicitly established by the 1930s, although it was used before, for instance, by Hilbert himself.[868] Frege's logical system encompasses 'more' than first-order mathematics; the axiom mentioned by Church resembles the expression used by Whitehead and Russell as the definition of identity in their *Principia Mathematica*, as we shall note below.[869] The last two 'Hilbertian-style' axioms are in essence those used in first-order languages when identity is considered as a primitive notion, but, to obtain the general case, *F*(*a*) must be admitted to be any formula whatever, while *F*(*b*) arises from *F*(*a*) from the substitution of some free occurrences of *a* by *b*, provided *b* is free for *a* in *F*(*a*) (*a* and *b* being individual variables).[870]

## 6.3.1 Identity in First-Order Classical Logic

Leaving aside the historical details, we will pay attention to the way classical first and higher order logics and set theory deal with identity. Suppose a first-order language. If it has only a finite number of predicates, then identity (in this sense) can be defined. For doing that it suffices to write as a formula $A(x, y)$ the conjunction of all possible substitutions in the predicates of the language, in a sense that there is an exhaustion of all the primitive predicates of the theory.[871] Then identity is defined by such a formula. For instance, suppose that the only primitive predicates are the binary predicate *P* and the unary predicate *Q*. Then $A(x, y)$ should be the following formula (except for the

[868] From Hilbert and Ackermann's *Grundzüge der Theoretischen Logik* (originally from 1928), we read that "[b]y an *axiom system of the first order* we understand one in which the individual axioms contain no predicate variables but only predicate constants. (…) If predicate variables also occur in the axioms, then we speak of an *axiom system of the second order*." (see Hilbert and Ackermann 1999, p. 107).

[869] Whitehead and Russell 1925. However, they restricted their definition to the so-called 'predicative properties' (predicative propositional functions), so that the application of identity in their system depends on the axiom of reducibility (ibid.). Russell had already used such a definition of identity in his Russell 1908.

[870] Mendelson 1997, p. 95.

[871] Cf. Hilbert and Ackermann op. cit., pp. 107 ff. It is worth noting that this was how Quine approached the issue.

quantifiers)

$$(P(xz) \leftrightarrow P(yz)) \wedge (P(zx) \leftrightarrow P(zy)) \wedge (Q(x) \leftrightarrow Q(y)) \tag{6.3.1}$$

which 'simulates' identity in the sense that $x$ and $y$ share all the primitive predicates of the language.

Furthermore, if the language has only a finite number of binary relations, and if they are the only predicate symbols, then we can postulate that all these relations are reflexive, symmetric and transitive and F2 above follows. (It is important to mention that F1 and F2 entail that the interpretation of ≈ is an equivalence relation).

These 'first-order' axioms have interesting consequences, in particular if we consider semantics. First, let us recall that F1 expresses the intuitive fact that 'every object is identical to itself', which is known as the Reflexive Law of Identity, and sometimes as one of the forms of the Principle of Identity, while F2 is the Substitutivity Law. If we intuitively think of identity as referring to something an object has to itself and with nothing else (the issue whether this 'something' is a legitimate 'property' or not will be touched on below), then we should expect that the semantical interpretation of the identity predicate ≈ should be the *diagonal* of the domain of discourse, namely, the set

$$\Delta_D =_{\mathrm{df}} (\langle x, y \rangle : x, y \in D \text{ and } x = y),$$

(6.3.2)

where $D$ stands for the domain of the interpretation. We remark that the symbol of identity in the *definiens* is the set-theoretical identity, for the interpretation of such a first-order language is made in a set-theoretical structure, according to the standard semantics. This is the main reason why we have used ≈ for identity *in the language* just to emphasize that the set-theoretical identity is a distinct (metatheoretical) concept.

A well-known result in standard logic says that by using F1 and F2 only, we cannot distinguish between 'individuals' of the domain and certain equivalence classes of individuals. In short, F1 and F2 do not 'characterize' the diagonal without ambiguity. To see why, suppose that $\mathscr{A} = \langle D, \varrho \rangle$ is a structure for our first-order language $L$ in the standard sense, where $\varrho$ is the denotation function, defined as usual, that is, such that for every individual constant $c$ of the language, $\varrho(c)$, which we denote by $c^D$, is an element of $D$; furthermore, for every $n$-ary predicate $R$, $\varrho(R) = R^D$ is a subset of the set $D^n$, the $n$th power of $D$, and for any $n$-ary functional symbol $g$, $\varrho(f)$ is a mapping from $D^n$ to $D$.

As we have seen above, the relation which interprets the primitive symbol of identity is in particular an equivalence relation (since it must be reflexive by F1 and symmetric and transitive, as can be easily proven).[872] Let us call $\approx_D$ such a relation. If $\mathscr{A}' = \langle D', \varrho' \rangle$ is another interpretation for our language such that its domain $D'$ is the quotient set of $D$ by the relation $\approx_D$ (that is, $D' = D/\approx_D$), and such that the relation which interprets the equality symbol in this new interpretation is denoted by $\approx_D$, let $f: D \to D'$ be the canonical mapping, which associates with every $x \in D$ its equivalence class $f(x) \in D'$ (that is, the equivalence class $f(x)$ to which $x$ belongs), defined as follows:

(i) $f(x) \approx_D' f(y)$ if and only if $x \approx_D y$
(ii) For every $n$-ary predicate letter $R$ of the language, if $\varrho(R) = R^D$ and $\varrho'(R) = R^{D'}$, then $R^{D'}(f(x_1), \ldots, f(x_n))$ if and only if $R^D(x_1, \ldots, x_n)$.
(iii) For every $n$-ary functional symbol $g$ of the language, if $\varrho(g) = g^D$ and $\varrho'(g) = g^{D'}$, then $g^{D'}(f(x_1), \ldots, f(x_n)) \approx_D f(g^D(x_1, \ldots, x_n))$.
(iv) For every individual constant $c$, we have that $c^{D'} \approx_D f(c^D)$

Then, we can prove that the structures $\mathscr{A}$ and $\mathscr{A}'$ are *elementarily equivalent*,[873] that is, for whatever sentence $\alpha(y_1, \ldots, y_n)$ we have that

$$\mathcal{A} \vDash \alpha(y_1 / x_1, \ldots, y_n / x_n) \text{ iff } \mathcal{A}' \vDash \alpha(y_1 / f(x_1), \ldots, y_n / f(x_n)).$$

(6.3.3)

In other words, $\alpha(y_1, \ldots, y_n)$ is true according to one interpretation if and only if it is true according to the other.[874] Of course, in order that this result be fully satisfied, it must be independent of the 'representative' $x_i$ we take as denoting the equivalence class $f(x_i)$, since if $x_k \approx_D x_i$, then both $x_k$ and $x_i$ give the same equivalent class, since $f(x_k) \approx_D f(x_i)$ in this case. What the equivalence between the structures says is that what holds for $x_1, \ldots, x_n$, holds also for $f(x_1), \ldots, f(x_n)$. As shown by Mendelson,[875] we can prove the existence of such an $f$.

Intuitively, this last result says that every element of the domain of $\mathscr{A}'$ (which is an equivalence class) acts as *an individual* of the domain of $\mathscr{A}$ as well. So,

[872] See Mendelson *loc. cit.*.

[873] Intuitively, the sentences of the considered language which are true in $\mathscr{A}$ are the same as those which are true in $\mathscr{A}'$ and conversely. In other words, two structures are elementarily equivalent if and only if they cannot be distinguished by means of the resources in $L$; see Mendelson, op. cit., p. 123.

[874] This result is proven with details in Hodges 1983.

[875] op. cit., p. 100.

from the point of view of the first-order language $L$, it is not possible to know whether we are dealing with an element of $D$ or with an equivalence class of $D'$ (hence, a certain collection of elements of $D$). For uniquely characterizing the diagonal $\Delta_D$, we need second order variables, that is, variables which run over collections of objects of $D$ or, alternatively (in extensional contexts), over their properties, but this will depend on the resources of the set-theoretical apparatus we are using for considering the structures, as we shall point out below.

Another way of seeing the limitations of first-order denumerable languages for characterizing identity may be as follows, here described in brief. First of all, we should agree that, since we cannot quantify over predicates, Leibniz's Law cannot be stated in full in first-order languages. But we could express something similar in the form of a schema, as follows: if $F$ denotes a predicate of individuals, then we could postulate that for every property $F$, $F(a)$ if and only if $F(b)$ entails $a \approx b$. In symbols, for any $F$,

$$(F(a) \leftrightarrow F(b)) \rightarrow a \approx b. \tag{6.3.4}$$

However, suppose that we have a domain $D$ with cardinality $\aleph_0$ and that our first-order language has at most a countable number of monadic predicates and, further, that $a$ and $b$ name two elements of $D$, which we denote respectively by $\varrho(a)$ and $\varrho(b)$, according to the above notation. As implied by the axioms of set theory (suppose Zermelo-Fraenkel set theory) where the interpretation is defined, $\varrho(a) = \varrho(b)$ if and only if for *every* subset $X$ of the domain, $\varrho(a) \in X$ if and only if $\varrho(b) \in X$. Since in standard semantics $F(a)$ is true if and only if $\varrho(a) \in F^D$ ($F^D$ is the extension of $F$), then the quantifier 'for all $F$' ranges over at most $\aleph_0$ subsets of $D$, while we know that $D$ (with such cardinality $\aleph_0$) has $2^{\aleph_0}$ subsets. Hence, even if (6.3.4) holds, this fact does not ensure that $\varrho(a)$ and $\varrho(b)$ are the very same element of $D$, for necessarily there exists a subset $Y \subseteq D$ which is not the extension of any predicate of the language such that $\varrho(a) \in Y$ but $\varrho(b) \notin Y$. Denumerable first-order languages also have limitations from this point of view.

## 6.3.2 Identity in Higher-Order Logic

In higher-order logics (given the language of the simple theory of types)[876] we can state a definition of the concept of identity which resembles Whitehead

[876] Hilbert and Ackermann 1950/1999.

and Russell's *Principia Mathematica*. Let us put it as follows:

$$a \approx b =_{\mathrm{df}} \forall F(F(a) \rightarrow F(b)), \tag{6.3.5}$$

with the usual restrictions regarding the types of the variables. Of course such a definition also involves a *type ambiguity*, in Russell's sense.[877] We also remark that Whitehead and Russell have used the material implication in the *definiens*, and not a biconditional in order to state Leibniz's Law. The reason for this, as noted by Boolos and Jeffrey,[878] is that we don't need the biconditional at the right, but there is a price to pay. Since this point is important for what follows, we shall reinterpret their argument here. First, let us consider an interpretation $\mathscr{A}$ such that $\mathscr{A} \vDash a \approx b$, that is, $\langle \varrho(a), \varrho(b) \rangle \in \Delta_D$, where $D$ is the domain of $\mathscr{A}$ (the notation is as above). Then, for every unary predicate $F$, $\mathscr{A} \vDash F(a) \rightarrow F(b)$, hence, by Generalization,[879] we have $\mathscr{A} \vDash \forall X\,(X(a) \rightarrow X(b))$. Conversely, let us assume that $\mathscr{A} \vDash \forall X\,(X(a) \rightarrow X(b))$. Hence, since $I_a(x) =_{\mathrm{df}} x \approx a$, then $\mathscr{A} \vDash I_a(a) \rightarrow I_a(b)$. But, since $\mathscr{A} \vDash I_a(a)$, as every object is identical to itself, so is $\mathscr{A} \vDash I_a(b)$. But then $\varrho(a) = \varrho(b)$ and therefore $\mathscr{A} \vDash a \approx b$.

As Boolos and Jeffrey say, (6.3.5) "is valid because among the properties of $a$ is the property of *being identical with a* [the predicate $I_a$ defined above]; then, if $b$ has to have *all* of $a$'s properties, it must have that one in particular".[880] This may seem to be an obvious remark, but it has been queried elsewhere whether the property 'being identical with $a$' is a legitimate (relational) attribute of $a$.[881] Thus, in considering these situations, we should pay attention to the way we state the definition of identity; without this 'problematic' property in the range of $X$, the definition *needs* the biconditional. This remark will have consequences for what follows, mainly with regard to the concept of non-rigid structures mentioned below.

But first let us emphasize once more the role played by set theory in proving that $a$ is identical to $b$, for this is what we intend to do when we (rightly) assert that a $\approx b$. For that, we shall sketch a minimum nucleus of the semantical counterpart of a second order logic (the argument of course can be generalized to general higher-order logics, but the second order case is suitable for our purposes) in order to see on what the intended interpretation of the defined

[877] This informally means that the definition is standing for an infinity of different definitions, one for each type.

[878] Boolos and Jeffrey 1989, p. 200.

[879] Recall that by Generalization we understand the inference rule of the predicate calculus (either of first or higher order) which states that $\forall x\alpha$ is derived from $\alpha$. See Mendelson op. cit., p. 70.

[880] *Loc. cit.*, their emphasis.

[881] Cf. the discussion in French 1989c.

symbol of identity depends. Second order logic is interpreted in a *second order structure*, which we may represent as

$$\mathcal{A} = \langle D, (R_i)_{i \in I}, \rho \rangle, \tag{6.3.6}$$

where

(i) $D \neq \varnothing$ is the domain of the individuals,
(ii) for each $i \in I$, $R_i$ is a non-empty set of $i$-placed relations on $D$; each element of $R_i$ is a subset of $D^i$.
(iii) $\varrho$ is a mapping which assigns to each individual constant $a$ of the language an individual $\varrho(a) \in D$, and to each $i$-place predicate constant $P$ an $i$-place relation $\varrho(P) \in R_i$.[882]

Now let $V$ be a mapping which assigns to each individual variable $x$ an element $V(x) \in D$ and to each $i$-place predicate variable $F$ an element $V(F) \in R_i$, which can be extended to a valuation $V^*$ so that $V^*(a) = \varrho(a)$ when $a$ is an individual constant and $V^*(P) = \varrho(P)$ if $P$ is a predicate constant. Such a mapping is called an *evaluation of variables*. If $\alpha$ is a formula of the language of our second order logic, then we can define as usual what it means for $\alpha$ *to hold* in $\mathcal{A}$ according to $V^*$,[883] which we express as

$$\mathcal{A}, V^* \models \alpha. \tag{6.3.7}$$

We are particularly interested in the case when $\alpha$ is the *definiens* of (6.3.5). In other words,

$$\mathcal{A}, V^* \models a \approx b \tag{6.3.8}$$

if and only if

$$\mathcal{A}, V^* \models \forall F \ \beta, \tag{6.3.9}$$

$\beta$ being the formula $F(u) \leftrightarrow F(v)$, where $a$ and $b$ are individual terms whatever and $F$ a predicate variable. As it is well known, (6.3.9) holds if and only if

$$\mathcal{A}, V'^* \models \beta \tag{6.3.10}$$

for every evaluation $V'^*$ such that $V'^*(t) = V^*(t)$ for all terms $t$ except probably $F$. But, this last condition holds if and only if

$$\mathcal{A}, V'^* \models F\left(u\right) \quad \text{iff} \quad \mathcal{A}, V'^* \models F\left(v\right), \tag{6.3.11}$$

[882] Enderton 1972, Chapter 4.

[883] See Robbin 1969, § 46 and Church 1956, § 54.

which holds if and only if both $V'^*(a)$ and $V'^*(b)$ (which are elements of $D$) belong to $V'^*(F)$, for every $F$, which are subsets of $D$.

Suppose that this last condition holds, that is, that we have proven that $V'^*(a)$ and $V'^*(b)$ belong to exactly the same subsets of the domain $D$ covered by the language we are considering. Let us emphasize this: in the semantical counterpart of our language, by means of the predicates it has as primitive, and of the defined interpretation, we *select* some subsets of $D$ which stand for the extensions of the predicates of the language, and not necessarily take *all* the subsets of $D$, for the interesting interpretations of second order languages (as in the case of higher-order languages in general) do not involve all the relations on the domain, according to the generalized Henkin-style semantics.[884]

However, even supposing that all these conditions are satisfied, can we ensure that $V'^*(a)$ and $V'^*(b)$ are the *same* element? In other words, if all these conditions are met, is (6.3.8) true? The answer is 'not necessarily', of course. In order to see why, we should insist that in item (ii) of definition (6.3.6), it was stated that for each $i \in I$, $R_i$ is a non-empty set of $i$-placed relations on $D$, and hence the above discussion depends on the relations chosen, in particular on those which interpret the unary predicates, which are being considered in the range of the variables $F$ above. As we have just emphasized, this does not entail that *all* subsets of the domain were taken into account. The consequences are that, for instance, if we take a domain $D = \{1, 2, 3, 4\}$ and if for interpreting the unary predicates of the language we choose some subsets of $D$, namely, the sets $\{1, 2\}$, $\{1, 2, 3\}$ and $\{1, 2, 4\}$, then since $V(a) = 1$ and $V(b) = 2$, of course $a \approx b$, for 1 and 2 belong to all these chosen subsets, although 1 is not *identical* with 2.

It is well known that we can repeat the procedure shown above for obtaining an interpretation which 'respects equality';[885] but even so this depends on the relations we have chosen to build the structure. The interpretations of $a$ and $b$ may belong to all the chosen subsets of the domain (and this can be extended to other higher types), but all we can prove is that $\mathscr{A} \vDash a \approx b$ for a certain $\mathscr{A}$, and this does not grant that the objects of the domain denoted by $a$ and $b$ are really and truly the very same object.

The only way to ensure that $a$ and $b$ are the same object is to look at the set-theoretical framework in which the second order structures are built; in other words, we need to be sure that $\forall \mathfrak{X} (a \in \mathfrak{X} \leftrightarrow b \in \mathfrak{X})$. There is no escape: we are strongly committed to set theory in semantic questions involving identity

[884] Church 1956, § 54.

[885] Robbin 1969, pp. 57 and 145.

(as in the case of other semantic questions, such as truth of course). Thus we shall look at some of its basic assumptions in the next section.

## 6.4 SET THEORY AND INDIVIDUATION

In his *Contributions to the Founding of the Theory of Transfinite Numbers*, Cantor stressed that

> [b]y an 'aggregate' (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) *M* of definite and separate objects *m* of our intuition or our thought. These objects are called the 'elements' of *M*.[886]

Bourbaki recalls that Cantor offered another 'definition' in the same vein in another work, saying that "[b]y a set is meant a gathering into one whole of objects which are quite distinct in our intuition or our thought".[887] According to Boolos, Cantor understood a set as a "many, which can be thought as one, i.e., a totality of definite elements that can be combined into a whole by a law".[888]

This intuitive characterization of *sets* (aggregates) captures the following distinctive features of such a conception:[889]

(i) The elements of a set are collected into a whole by a certain 'law'.
(ii) The set is 'determined' by its elements.
(iii) The elements of a set must be *distinct* from each other.
(iv) The elements of a set are in some way *given before* the set properly.

These four characteristics suggest four basic principles underpinning the concept of set, namely:[890]

(i) The Principle of Comprehension (or Abstraction).
(ii) The Principle (or Axiom) of Extensionality.
(iii) A concept of identity for the elements of a set.
(iv) The iterative concept of set.

[886] Cantor 1955, p. 85.

[887] Bourbaki 1994, p. 25. The translation is from Cantor 1932, p. 282.

[888] Boolos 1964.

[889] Similar points are mentioned in Casari's analysis of the Cantorian concept of set; see Casari 1976, pp. 21 ff, where similar considerations are given.

[890] Casari *loc. cit.* offers others, such as the 'absoluteness' of the concept of set in the sense of its independence of the language; but the selected characteristics suffice for our purposes here.

Let us consider each in turn. First, the Principle of Comprehension may be written, in a language obviously not used in Cantor's time, as follows: given a certain property $P$,

$$\exists y \, \forall x (x \in y \leftrightarrow P(x)).$$

(6.4.1)

This 'set' may be written, as usual, as $y = \{x\colon P(x)\}$, where ':' is read 'such that'. It is a well-known fact that this principle, as just stated, leads to a contradiction, as Russell's paradox shows. For instance, take $R$ to be the property such that $x$ satisfies $R$ if and only if $x$ is a collection of objects which does not belong to itself (in symbols, $R(x) \leftrightarrow x \notin x$). Thus, the collection of all cats has such a property, for being the collection of all cats, it is obviously not a cat, hence it does not belong to itself. So, by the above principle we can form the following set: $\mathscr{R} = \{x\colon R(x)\}$, and then, as is easy to see, $\mathscr{R} \in \mathscr{R}$ if and only if $\mathscr{R} \notin \mathscr{R}$.

Zermelo's axiomatic version of set theory overcomes this kind of problem, and the above principle has been substituted by a schema, known as the Separation Axiom, which states that

$$\forall z \exists y ( \forall x (x \in y \leftrightarrow x \in z \wedge P(x))),$$

(6.4.2)

where $P$ is (in Zermelo's words, adapting the notation to that used here) "a propositional function defined for all elements of $z$".[891] The problem regarding the correct characterization of such a $P$ has led to several important discussions, culminating in Skolem's formulation of set theory in first-order language; in this vein, $P(x)$ is a well-formed formula of the first-order predicate calculus.[892]

The nature of the 'property' $P$ was not made clear by Cantor, as just mentioned, but in trying to give some hints, he helps us in further understanding his conception of set. For instance, he says that

> a variety (an aggregate, a set) of elements that belong to a certain conceptual subject is well defined if by virtue of its definition and of the Principle of the Excluded Middle it must be determined as *internally determined* whether an element of such a conceptual subject is an element of the variety, *so as* if two objects belonging to the set, despite the formal diversity by means of which they are given, are identical or not. (…) In general, the relative decision cannot be performed with security and exactness by means of the methods and capacities at our disposal, but this is not relevant. What is relevant is only their *internal decision* which, in the concrete cases, where it is required,

[891] Zermelo 1908.

[892] van Heijenoort's anthology reproduces English translations of all these papers; further information can be found also in Fraenkel, Bar-Hillel and Levy 1973.

must be transformed in an *actual decision* (external) by means of an improvement of the tools.[893]

Thus we see that the Comprehension Principle embodies an important additional hypothesis, namely, that any object whatever can have properties, and hence can be a member of a set.[894] But the above quotation seems to suggest more; that is, Cantor is apparently suggesting that it is by means of 'the laws of logic' (although he mentions only the Principle of the Excluded Middle, the other principles of what was later called 'classical logic' seem to be implicit) that we should verify whether a certain object is or is not an element of a set, and, further, that sometimes this fact cannot be actually determined.[895]

The second item above, namely, the Extensionality Principle, is clear: a set is *determined* by its elements, and 'two' sets are identical (that is, they are the very same set) if and only if they have the same elements. But of course this depends on a theory of identity for the elements of a set, which is our third selected item, to which we shall pay special attention below, where items (ii) and (iii) are considered in more detail. The fourth one makes reference to the iterative concept of set. One can discuss whether this conception, which roughly says that a set can be 'formed' only *after* its elements have been formed (the notion of 'after' here is to be taken in the same sense as the Bibliography of this book is written *after* its Introduction) is implicit or not in Cantor's assumptions. Such exegesis does not interest us here, but it seems clear that the intuitive conception of set presupposes that sets are 'well founded'.[896]

The crucial question now is, how to treat indistinguishability in a framework built to talk of individuals? In the next section we shall consider some of the ways this concept has been taken into account in mathematical contexts.

## 6.5 CHARACTERIZING INDISTINGUISHABILITY

An informal characterization of the concept of set, which came from Cantor, as we have seen, states in brief that a set is a collection of *distinct*, or *discernible*, objects. We have also emphasized that the axiomatic versions of set theory

[893] *Apud* E. Casari, op. cit., p. 22.

[894] As also remarked by Casari, op. cit., p. 23.

[895] It should be recalled that the rudiments of recursion theory tell us that there is no decision method for the verification of whether a certain $x$ is or is not an element of a set $y$, that is, there is no algorithm which, given $x$ and $y$ as inputs, gives a decision whether $x$ belongs to $y$ or not, unless we particularize $x$ and $y$.

[896] A detailed analysis on the iterative conception of set is given in Boolos' paper mentioned above, and also in other papers of the same nature in Benacerraf and Putnam's book already cited.

(Zermelo-Fraenkel, von Neumann-Bernays-Gödel etc.), despite their differences, all 'accept' this state of affairs in a certain sense (mainly if we consider also the Axiom of Regularity).[897] As we have noted, in order for the Axiom of Extensionality to hold, it is necessary to have a criterion for asserting whether two elements are the same object or not. In other words, standard set theories presuppose a theory of identity for both the elements of a set and for the sets themselves (generally given by the underlying first-order classical logic *plus* the Axiom of Extensionality).

According to this theory, as we have seen, there is no place for 'indistinguishable' objects, that is, for entities which differ *solo numero*: if they differ, there exists a set—which corresponds to a property—to which one of them belongs while the another one does not. In this section we shall discuss this point in some detail. But before we do that, let us recall some of the ways mathematicians have found for talking about indistinguishable things in such standard contexts. As we shall point out, all such ways amount to 'mathematical tricks' in *considering* that some objects can be *thought of* as if they were indistinguishable, while at bottom remaining distinct objects.

## 6.5.1 Weyl's Strategy

As already noted in previous chapters, in considering 'aggregates of individuals' for discussions on the foundations of quantum theory, Weyl aimed to treat the case where the elements of a certain collection may be in certain 'states' but only the quantity of them in each of these states could be known.[898] This is of course what happens in quantum physics, and provides an interesting parallel to what happens in general. Thus, it will be illustrative to have a look at Weyl's work. As he says,

> [i]n physics one aims at making division into classes so fine that no refinement is possible; in other words, one aims at a *complete* description of state. Two individuals in the same 'complete state' are indiscernible by any intrinsic characters—although they may not be the same thing.[899]

To represent this idea, Weyl considers a *set*$S$ (let us emphasize that $S$ is a *set*, hence a collection of *distinct* objects) with $n$ elements, say $x_{p_1}, \ldots, x_{p_n}$,

[897] Roughly speaking, this axiom prevents a set from belonging to itself, as in the case of circular sets, like those $x$ such that $x \in x_1 \in x_2 \in \ldots \in x_n \in x$ and infinite descending chains $\ldots x_n \in x_{n-1} \in \ldots \in x$. In the well-founded universe, the content of a set always stops in some set (or in the *Urelemente*). See Fraenkel *et al.*1973.

[898] Weyl 1949/1968, App. B.

[899] op. cit., p. 245.

endowed with an equivalence relation ~. The intuitive interpretation is that $a \sim b$ means that $a$ and $b$ are of the same *kind*, or *nature*, and in this case they are said to belong to the same *state*. The equivalence classes $C_1, \ldots, C_k$ of the quotient set $S/\sim$ stand for these 'states' in Weyl's account. An *aggregate*$S$ is "a set of elements each of which is in a definite state; hence, the term aggregate is used in the sense of 'set of elements with an equivalence relation'".[900] So, an aggregate is a pair $\langle S, \sim \rangle$, where ~ is an equivalence relation on the non-empty set $S$. A certain *individual state* of the aggregate is then achieved when "it is known, for each of the $n$ marks $p$, to which of the $k$ classes the element marked $p$ belongs". If the elements of $S$ are distinct from one another, then of course there are $k^n$ possible individual states of the aggregate, but, as Weyl remarked,

> [i]f, however, no artificial differences between elements are introduced by the labels $p$ and merely the intrinsic differences of state are made use of, then the aggregate is completely characterized by assigning to each class $C_i$ ($i = 1, \ldots, k$) the number $n_i$ of elements of $S$ that belong to each class $C_i$. These numbers, the sum of which equals $n$, describe what may conveniently be called the *visible or effective state* of the system $S$. Each individual state of the system is connected with an effective state, and any two individual states are connected with the same effective state if and only if one may be carried into the other by a permutation of the labels.[901]

In other words, since each equivalence class has a cardinal $n_i$, $i = 1, \ldots, k$, the effective state of the aggregate is characterized by the *ordered decomposition* $n_1 + \cdots + n_k = n$. Then, if the individuality of the elements of $S$ is forgotten for a moment and only this ordered decomposition is considered, we arrive at a formula which expresses the number of different effective states, which is the formula for Bose-Einstein statistics, given in Chapter 3, namely,

$$\frac{(n+k-1)!}{n!(k-1)!}. \tag{6.5.1}$$

Although adequate for mathematical purposes, Weyl's suggestion of considering a set endowed with an equivalence relation does not deal with the indistinguishability *solo numero* of the elements of $S$, but in effect only mimics indistinguishability. Of course, in order to arrive at the above formula, that is, at the situation where only the permutation of the labels is given (according to the above quotation), one has to suppose that certain elements of some class $C_i$ were permuted with elements of a class $C_j$ ($i \neq j$) in such a way that their

[900] op. cit., p. 239.

[901] op. cit., pp. 239–40.

cardinalities are preserved, so that the ordered decomposition stays the same. But the permutation of distinct elements of course changes the classes, due to the Axiom of Extensionality (see below); in other words, after the permutation, the 'states', viewed as sets, are no longer the same! For the mathematical description of physics, perhaps this is not important, for it satisfies what Weyl called the Principle of Relativity, according to which "[o]nly relations and statements [we should say, physical laws] have objective significance as are not affected by any change in the choice of the labels *p*".[902]

This line of thought is the basis for the use of group theory in quantum mechanics (cf. this volume, p. 142), and we should recall that Weyl was one of the founders of this application.[903] But, philosophically, we require a more precise mathematical mechanism to consider not only that the permutations of the particles (the above 'change of labels') do not change physical laws, but also to express the fact that

> [a]ccording to quantum mechanics, any two electrons must necessarily be completely identical, and the same holds for any two protons and for any two particles whatever, of any particular kind. This is not merely to say that there is no way of telling the particles apart; the statement is considerably stronger than that. If an electron in a person's brain were to be exchanged with an electron in a brick, then the state of the system would be *exactly the same state* as it was before, not merely indistinguishable from it! The same holds for protons and for any other kind of particle, and for the whole atoms, molecules, etc. If the entire material content of a person were to be exchanged with the corresponding particles in the bricks of his house then, in a strong sense, nothing would have happened whatsoever. What distinguishes the person from his house is the *pattern* of how his constituents are arranged, not the individuality of the constituents themselves.[904]

Penrose's quotation emphasizes the important role of permutational symmetries in physics, which we should try to consider 'set-theoretically', as Weyl did. In taking a set to begin with, Weyl could not consider what we call 'legitimate indistinguishable objects', as follows from Cantor's 'definition' above, and this is so independently of the particular *standard* set theory we use. So, the (in principle) 'identifiable' characteristics of the elements of *S* in Weyl's treatment of the problem were masked by a trick of 'forgetting' that they are

[902] op. cit., p. 240. The effective state of an aggregate helps in stating that "the principle of relativity finds expression in the postulate of invariance with respect to the group of all permutations" (ibid.) This principle, or its equivalent, may form a fundamental component of structuralist views of physics.

[903] Another being Wigner, as is well known; see Bueno and French 1999.

[904] Penrose 1989, p. 32.

elements of a set, and only their role as elements of an equivalence class was taken into account. This of course may do the job in physical terms, but from the philosophical point of view another kind of answer (set-theoretical) should be looked for, as Manin emphasized. We shall be more explicit on this point in the next section.

## 6.5.2 Indiscernibility and Structures[905]

In the context of standard set theories, such as those mentioned previously, it is possible to characterize a notion of indistinguishability, for instance by considering the idea of invariance under automorphisms of a structure.[906] As is well known, the idea of invariance under automorphisms was emphasized by Klein in his classification of the different geometries by means of invariance under the transformations of certain groups. We shall see that this general idea can also be helpful here. The main idea is to call 'two' individuals $a$ and $b$ indistinguishable (or indiscernible) in relation to a certain structure if and only if they share all the absolutely definable properties of this structure, that is (in set-theoretical terms), when they belong to the same collections of elements of the domain that are invariant under automorphisms.[907]

More precisely, let $\mathfrak{A}$ be a structure.[908] Then we say that $a$ and $b$ (which belong to the domain of the structure) are $\mathfrak{A}$-*distinguishable* (or distinguishable in the structure $\mathfrak{A}$) if and only if there exists a sub-collection $X \subseteq D$ such that: (i) $X$ is invariant under automorphisms of $\mathfrak{A}$, that is, $f(X) = X$ for every automorphism $f$ of $\mathfrak{A}$, and (ii) $a \in X$ if and only if

[905] The concept of indistinguishability in a structure presented in this section was explored by Antonio Coelho (Coelho 2000); this section is based on Krause and Coelho 2005.

[906] Roughly, an automorphism is a bijective mapping on the domain of the structure which 'preserves' all the relations of the structure; see the examples below.

[907] In this discussion, we shall restrict ourselves to considering 'first order structures', which of course are not sufficient for discussing physics. But the ideas we present can be extended to more general mathematical structures, as shown by Newton da Costa in his forthcoming paper, although this extension demands a more detailed mathematical development. Here, we shall focus on the general ideas only.

[908] Here, by a structure we mean a set-theoretical object like $\mathfrak{A} = \langle D, \{R_i\} \rangle_{i \in I}$, where $D$ is a non-empty set (for applications in physics, $D$ could be obtained from other 'basic' sets by the usual set-theoretical operations, in a process which resembles that of Bourbaki (1968, Chapter 4)), and $R_i$ are relations on $D$ (an operation of rank $n$ is a relation of rank $n + 1$).

$b \notin X$. Otherwise, we say that $a$ and $b$ are $\mathfrak{A}$-*indistinguishable*. So, in saying that $a$ and $b$ are $\mathfrak{A}$-indistinguishable, we mean that for every sub-collection $X \subseteq D$, if $X$ is invariant under automorphisms of $\mathfrak{A}$, then $a \in X$ if and only if $b \in X$. Equivalently, we may say that $a$ and $b$ are $\mathfrak{A}$-indistinguishable if and only if there exists an automorphism $f$ of the structure $\mathfrak{A}$ such that $f(a) = b$.

A typical example is the following. Let us take the structure $\mathfrak{G} = \langle Z, + \rangle$ (the additive group of the integers). Then the only automorphisms of $\mathfrak{G}$ are the identity function and the function $f$: $Z \mapsto Z$ defined by $f(x) = -x$.[909] In this case, 4 and $-4$ are $\mathfrak{G}$-indistinguishable, as are all integers $x$ and $-x$. In other words, *within the structure* we cannot distinguish between, say, 4 and $-4$. But of course 4 and $-4$ are not the same integers (they are not *identical*), although the differences (given by relevant properties) can be seen only *from outside* the structure. For instance, we can define a property $F$, which does not belong to the structure, meaning 'to be greater than 0', possessed by either $x$ or $-x$, but not both (supposing $x \neq 0$).

Then, the question arises: given a structure $\mathfrak{A}$, if there are elements of its domain which are $\mathfrak{A}$-indistinguishable, can we find adequate properties (outside of the structure) which distinguish them? We shall answer this question for a restricted case which interests us here: we shall show that, within standard set-theoretical contexts, we can always find such relations. In other words, within the usual set-theories, indistinguishability can be considered only in relation to a certain structure, but there is no indistinguishability *tout court*. That is, the objects treated by standard set theories (like Zermelo-Fraenkel with regularity) are *individuals*, in accordance with Cantor's intuitive concept of a set.

To put things clearly, let us introduce the following definition. We say that a structure $\mathfrak{A}$ is *rigid* if and only if its only automorphism is the identity function. For instance, if $\mathfrak{A} = \langle D, < \rangle$, where $D$ is a non-empty set and $<$ is a well ordering on $D$, then $\mathfrak{A}$ is rigid.[910] In a rigid structure, there are no distinct elements which are $\mathfrak{A}$-indistinguishable (as in the above example where 4 and $-4$, although distinct, are $\mathfrak{G}$-indistinguishable). Conversely, it is easy to prove that

[909] We recall that in this case the automorphisms of the structure $\mathfrak{G}$ are those bijective mappings $h$: $Z \mapsto Z$ such that $h(x + y) = h(x) + h(y)$.

[910] By a well-ordering on $D$ we mean a linear ordering on $D$ (that is, a transitive relation which obeys trichotomy, that is, for any $x$ and $y$ in $D$, exactly one of the following alternatives holds: $x < y$, $x = y$ or $y < x$) such that every non-empty subset of $D$ has a least element ($a$ is a least element of $X$ iff $a < x$ for all $x$ in $X$). That the only automorphism of A is the identity function can be seen as follows. By reduction, suppose that $f$ is an automorphism of A which is not the identity function. Then there exists a least element $a \in D$ such that $f(a) \neq a$. But we can't have $f(a) = b < a$, since for every $b < a$, it follows that $f(b) = b$, given that $a$ is the least element such that $f(a) \neq a$. Then $f(a) > a$ but, since $f$ is surjective, there exists $c \in D$ such that $f(c) = a$. But neither $c = a$, for $f(a) \neq a$, nor $c < a$, for $f(b) = b$ for every $b < a$. Hence $c > a$ and $f(a) > f(c)$, which contradicts $f$ being an automorphism.

if $\mathfrak{A}$ is a structure where $\mathfrak{A}$-indistinguishability and identity coincide, that is, *a* and *b* are $\mathfrak{A}$-indistinguishable if and only if $a = b$, then $\mathfrak{A}$ is rigid. Suppose that *f* is an automorphism of $\mathfrak{A}$ which is not the identity function. Then there exists *a* in the domain such that $f(a) = b \neq a$. But, since $b \neq a$ and in such a structure by hypothesis identity and $\mathfrak{A}$-indistinguishability coincide, then there exists a sub-collection $X$ of the domain such that: (i) $X$ is invariant under automorphisms; (ii) $a \in X$ but $b \notin X$. But this is a contradiction, for as $X$ is invariant under automorphisms and $a \in X$, we should have $f(a) = b \in X$. An important example of a rigid 'structure' is $\mathfrak{V} = \langle V, \in \rangle$, where $V$ is the usual well-founded universe of ZF (the class of all well-founded sets).[911]

Thus, in a rigid structure, the sets of all $\mathfrak{A}$-indistinguishable objects and the set of all identical objects (the diagonal of the domain) coincide. This enables us to say that in a rigid structure, all elements of the domain are individuals and, in such structures, we may use the property 'to be identical with *a*' for characterizing *a* and for distinguishing it from other objects of the domain. The above problem can now be expressed in the following terms: can we make a certain structure rigid? By 'making rigid' we mean completing the collection of the relations of the structure with other relations so that we can 'individuate' all elements of the domain. A simple example is the following. Suppose again the additive group of the integers, that is, the structure $\mathfrak{G} = \langle \mathrm{Z}, + \rangle$, which is not rigid, as we have seen above. It is easy to extend (or 'to expand') $\mathfrak{G}$ to a structure $\mathfrak{G}'$ which is rigid, simply by adding to the structure all the singletons of the elements of Z, that is, by considering the structure $\mathfrak{G}' = \langle \mathrm{Z}, +, \{0\}, \{1\}, \{-1\}, \ldots \rangle$ (it should be noted that this corresponds to introducing in the structure the properties—or predicates—'to be identical with 0', 'to be identical with 1' etc.).

This way of 'completing' with singletons the relations in the structure to make it rigid is not the only possibility. Let us say first that if $\mathfrak{A} = \langle D, \{R_i\}_{i \in I} \rangle$ is a structure, then a structure $\mathfrak{B}$ is an *expansion* of $\mathfrak{A}$ if and only if $\mathfrak{B} = \langle D, \{R_i\}_{i \in I \cup J} \rangle$, where $I \cap J = \varnothing$. In other words, $\mathfrak{B}$ is an expansion of $\mathfrak{A}$ if and only if $\mathfrak{B}$ is obtained by adding new relations to $\mathfrak{A}$. For example, $\langle \mathrm{Z}, +, < \rangle$ is an expansion of $\langle \mathrm{Z}, + \rangle$. We say that $\mathfrak{B}$ is a *trivial rigid expansion* of $\mathfrak{A}$ if and only if the following clauses are met: (i) $\mathfrak{B}$ is an expansion of $\mathfrak{A}$; (ii) $\mathfrak{B}' = \langle D, \{R_i\}_{i \in J} \rangle$ is rigid (in this case, $\mathfrak{B}$ is also rigid). This means that the

[911] The word 'structure' is in quotation marks in order to call our attention to the fact that $V$ is not a *set* of ZF. But, even so, we can talk of automorphisms of $\langle V, \in \rangle$.

new relations added to $\mathfrak{A}$ in order to obtain $\mathfrak{B}$ are, alone, sufficient to ensure the rigidity of $\mathfrak{B}$ regardless of the original relations of $\mathfrak{A}$. The above example of taking $\mathfrak{G} = \langle Z,+\rangle$ and its expansion $\mathfrak{G}' = \langle Z,+,\{0\},\{1\},\{-1\},\ldots\rangle$ shows what a trivial rigid expansion means (we remark that $\mathfrak{B}' = \langle Z,\{0\},\{1\},\{-1\},\ldots\rangle$ is obviously rigid). This example also shows that every structure has a trivial rigid expansion, which can be obtained simply by adding to it all the singletons of the elements of its domain.

We also say that a structure $\mathfrak{B}$ is a *non-trivial rigid expansion* of $\mathfrak{A} = \langle D,\{R_i\}_{i\in I}\rangle$ if and only if (i) $\mathfrak{B}$ is an expansion of $\mathfrak{A}$, (ii) $\mathfrak{B}$ is rigid, but (iii) $\mathfrak{B}' = \langle D,\{R_i\}_{i\in J}\rangle$ is not rigid. In this case, the new relations added to $\mathfrak{A}$ are not, *alone*, sufficient to ensure the rigidity of $\mathfrak{B}$ regardless of the original relations of $\mathfrak{A}$. For example, if $\mathfrak{G} = \langle Z,+\rangle$, then $\mathfrak{G}'' = \langle Z,+,<\rangle$ is a non-trivial rigid expansion of $\mathfrak{A}$, since $\langle Z, +, <\rangle$ is rigid because the only automorphism of $\langle Z, +\rangle$ distinct from the identity function is $f$: $Z \mapsto Z$ defined by $f(x) = -x$, which does not preserve the order $<$. But $\langle Z, <\rangle$ is not rigid, since, for each $k \in Z$, the function $f_k$: $Z \mapsto Z$ defined by $f_k(x) = x + k$ is an automorphism of $\langle Z, <\mathrm{ar}\rangle$. As another example, take again $\mathfrak{G} = \langle Z,+\rangle$. Then $\mathfrak{G}^* = \langle Z,+,\{k\}\rangle$, where $k \in Z$ is a fixed but arbitrary non-zero element of Z, is a non-trivial rigid expansion of $\mathfrak{G}$, for although $\mathfrak{G}^*$ is rigid (again, the only automorphism of $\langle Z, +\rangle$ distinct from the identity function does not preserve $\{k\}$ for $0 \neq k \in Z$), $\langle Z, \{k\}\rangle$ is not, since any permutation of Z that fixes $k$ is an automorphism of $\langle Z, \{k\}\rangle$. On the other hand, $\langle Z, +, \{0\}\rangle$ is not a non-trivial rigid expansion of $\langle Z, +\rangle$, because it is not rigid, for the only automorphism of $\langle Z, +\rangle$ which is not the identity function being $f(x) = -x$, then $f(k)$ will be $-k \neq k$ if $k \neq 0$. These definitions are not arbitrary; on the contrary, they can help us in discussing certain philosophical issues regarding what could be a 'structure' for quantum theory. Let us now touch on this point in brief.

## 6.5.3 The Implications for the Philosophy of Quantum Theory

The main lesson we can learn from the above discussion is that within standard set theoretical frameworks—which means classical mathematics—we can deal with indiscernibility (or indistinguishability) only in relation to a certain structure. Of course when it comes to quantum physics, the considered structures would be higher-order structures, but the above discussion can be generalized to cover them, so we shall continue to use our examples given previously.

This means that in order to consider objects *a* and *b* as indistinguishable, we need to 'forget' certain distinctive properties they may have and consider only those which are relevant for the purposes we have in mind, but which are not sufficient for them to be regarded as distinguishable individuals. But, since any structure built in a theory like ZF can always be expanded (although perhaps not trivially) to the rigid structure $\langle V, \in \rangle$, then 'from the outside' of the considered structure, that is, from the point of view of this whole well-founded universe $\langle V, \in \rangle$, all objects (sets and eventually the *Urelemente*) do have individuality, in the sense of being distinct from any other element at least by the property 'being identical with *a*' (which in extensional contexts corresponds to the singletons $\{a\}$), which individuates *a*; formally, we can define such a property as we have done before, say by positing $P_{\hat{x}}(x) =_{\text{def}} a = a$ (or by adding to the structure the singletons of the elements of the domain).

This way of considering indistinguishability only with respect to a structure is equivalent to what we have called 'Weyl's strategy'. We recall that Weyl proposed (not explicitly, of course) 'to forget' that the elements of *S* were elements of a *set* (hence being individuals in $\langle V, \in \rangle$) and considered only the equivalence classes to be the 'states' the elements are in. In a certain sense, when we select the symmetric and antisymmetric vectors in a Hilbert space or the symmetric and antisymmetric solutions of Schrödinger's equation to be the relevant vectors or solutions, we are doing the same; that is, we are restricting our discourse to an adequate 'structure' where everything happens as if there are indistinguishable objects. But, if the vectors (or the solutions of the Schrödinger's equation) are to stand for collections of quanta, as when we say that a certain vector 'represents' a collection of *n* 'identical' particles, this should be viewed as a mathematical device for hiding the information that they are individuals, for, as elements of a set, the entities these vectors or solutions represent *can* be individuated. In standard mathematics, since any structure can be expanded to a rigid structure, there are no legitimate non-individuals, and the Received View can be dealt with only by adding an additional hypothesis which entails that the individuality characteristic of the entities will not be taken into account.

We have emphasized above that Schrödinger called our attention to the limitations of the language of classical physics for dealing with quantum entities (see this volume, 246). Perhaps it is precisely the use of 'standard languages' (read: classical mathematics built within a standard set theory), where every

object is an individual, which makes a hypothesis like the Indistinguishability Postulate (or the Symmetrization Postulate) necessary. Perhaps a true quantum mechanical language[912] should not require such postulates. Standard set-theoretical languages create a 'land of individuals', and the use of labels (as the coordinates of the particles) for writing expressions like Schrödinger's equation, imposes the necessity of symmetrization hypotheses. In a certain sense, Redhead and Teller have acknowledged this, when they say that "[t]he labels [imposed by the needs of a treatment given within standard mathematics], together with particle indistinguishability [that is, with the need for considering indistinguishability], create the need for symmetrization or antisymmetrization (or, in principle, higher-order symmetries)".[913]

The above defined concept of indistinguishability in a structure also helps us in clarifying the two metaphysical alternatives we introduced earlier. We recall again that, according to the 'Received View', quantum particles cannot be regarded as on a par with 'macroscopic' objects like rocks and people, and that they are, in a sense, 'non-individuals'. One way of understanding this idea is to claim that identity does not make sense for such particles, as Schrödinger has emphasized. However, as we have seen, this view does not necessarily follow from the physics and one could maintain that, on the contrary, quantum particles can, in fact, be regarded as individuals, albeit with very different properties and behaviour from their classical counterparts. Hence, as we have said, our fundamental metaphysics is underdetermined by our physics.

Taking into account what we have discussed in this section, we can say that those who adopt the Received View, so accepting non-individuals, would have to agree that a suitable structure for quantum mechanics (whatever this means) cannot have a non-trivial rigid expansion, which intuitively means that the rigidity of a structure for quantum theory can be achieved only by considering new relations which, by themselves, regardless of the quantum nature of the elements of the domain, guarantee such a rigidity. This would be true in particular for those who reject Weyl's strategy, and would like to consider Post's view that non-individuality should be attributed to quanta "right at the start". In other words, the metaphysical position according to which elementary particles are non-individuals can be sustained only in those structures that cannot be made rigid by adding new objects, relations and functions to the

[912] We recall that Manin has suggested that quantum mechanics does not have its 'own' language; see this volume, 240.

[913] Redhead and Teller 1991. In the next chapter, we shall see how a suitable mathematical language enables us to obtain symmetric situations without *ad hoc* postulates.

structure in order to make it rigid. Taking into account what was said above, namely, that a structure can be made rigid by adding the singletons of the elements of its domain or, equivalently (in extensional contexts) the properties 'being identical with *a*', where *a* ranges over the domain of the structure, the Received View of quantum entities can be realized only if we (at least) drop such properties from the pantheon of the attributes of a particle. This has obvious consequences in considering Leibniz's Law, as we have seen before. But, since in set-theoretical terms (hence, we may say, in mathematical terms) the existence of the singleton of *a* corresponds to the existence of an identity criterion for *a*, since for whatever *b*, we may say that $b = a$ if and only if *b* belongs to the singleton of *a* and conversely, the Received View is linked to a view where the predicate of identity should not be applied to the entities considered. In short, in its foundations, the Received View seems to require the failure of identity criteria, and this of course does not entail that we do not have 'entities' of a sort. The labels that are introduced (as in the first quantized approach) play the role of singletons, for the nature, classical or quantum, of the objects does not matter from the point of view of using labels. The only relevant point is that the labels provide a way of associating a well-ordered structure, that is, an ordinal, to the collection of the objects considered. But, in considering 'legitimate' indistinguishable quanta, no such possibility can be achieved, even in principle; so, in order to consider non-individual entities 'right at the start', we must not be restricted either to discussions *within* a structure or within standard set theories.

The alternative metaphysical view (which considers quanta as individuals), however, is consistent with a position according to which the structure $\mathfrak{A}$ has a non-trivial rigid expansion. That is, in this case we accept that the objects of the domain (including elementary particles) *may* be distinguished, but not merely by posing new relations such as well-orders, that is, labels that by themselves are, *alone*, responsible for such distinguishability. In other words, in considering individuals, we accept that the objects of the domain (elementary particles, say) *can* be distinguished from each other. The restriction to a certain structure (or to a context where certain properties are not taken into account) makes sense of the idea of indistinguishability, but this is also from the point of view of the considered structure (or context). To push the discussion a little further, suppose we have found a language where no identity conditions can be dealt with (we shall consider such an example in the next chapter). How can we maintain individuality in such a context? One answer is to proceed by introducing an alternative 'set' theory.

Finally, we would like to emphasize that we should not ignore the power and limitations of the underlying logical and mathematical apparatus in the philosophical discussions of individuality. Sometimes the problems cannot be appropriately analysed only at the informal level, without taking into consideration the logical and mathematical axioms of levels (1) and (2) respectively of a theory. A 'more mathematical' consideration may illuminate some of the problems, as the concept of rigid structures does, according to the above guidelines. This kind of discussion may help in making our philosophical 'intuitions' more precise. Thus, for example, the development of the idea of rigid structures helps illuminate the status of the claim that 'to be identical with *a*' is not a legitimate relational property of *a*.

In the next chapter, we shall examine a 'set'-theory which (we suggest) directly responds to Manin's problem.

# 7
# The Mathematics of Non-Individuality

If mathematical thinking is defective, where are we to find truth and certitude?

D. Hilbert, *On the Infinite*, 1925

As we saw in Chapter 5, Dalla Chiara and Toraldo di Francia have proposed that *quaset theory* might provide the mathematical tools for the semantic analysis of the language of microphysics. According to them, standard set theories are not adequate to represent microphysical phenomena, for the sorts of reasons we have already considered. Furthermore, they have suggested that issues of identity in quantum theory demand a kind of *intensional semantics*, for which quaset theory is supposed to provide an appropriate (meta)mathematical framework.

Motivated by distinct yet related reasons, da Costa discussed the possibility of presenting logical systems in which some form of the Principle of Identity is restricted. His motivations were essentially philosophical, in trying to show that the laws of classical logic are not so secure that they cannot be violated.[914] Based on Schrödinger's idea that the concept of identity lacks sense in the quantum context, da Costa defined a two-sorted first-order logic in which identity statements such as $a = b$ make sense only with respect to the objects of one of the sorts considered; for the others (which might be regarded as denoting quantum particles), the expression $x = y$ simply is not a formula. Hence, for these latter objects, it is not possible to say (within the theory) either that they are identical or that they are distinct from one another.

[914] da Costa 1980, pp. 117–20; 1997, pp. 123–6.

# 7.1 THE NAME OF THE GAME

Da Costa realized that a complete semantics could be stated for these 'Schrödinger Logics', but he noted that such a semantics, if grounded in the standard set theories, like ZF, would not be adequate to express the intuitive idea of collections of objects for which the concept of identity does not apply.[915] Thus he proposed that a kind of theory of *quasi sets* should be developed, in which standard sets were to be viewed as particular cases and, then, it was suggested, in such a theory a more adequate semantics for his logics could be obtained, although da Costa did not indicate how this might be developed.

In 1990, a form of *quasi set theory* was proposed to develop these ideas, and subsequently it has been improved in certain respects.[916] The main motivation was not only to obtain a mathematical framework in terms of which the semantics for Schrödinger logics could be provided, but also to pursue Schrödinger's intuitions and to explore the mathematical counterpart of a theory which admits collections of objects for which identity and diversity are meaningless concepts. However, it was determined that this should be done in such a way that, taking into account the motivation provided by the quantum mechanical treatment of elementary particles, a weaker concept of 'indistinguishability' could be considered as holding among certain elements.[917]

The importance of the development of such a mathematical framework can also be appreciated from the following perspective. Since the 1950s, Suppes has defended the claim that "to axiomatize a theory is to define a set theoretical predicate".[918] This summarizes the fact that practically every concept of standard mathematics or even of empirical sciences, like physics, can be formulated (or expressed) within the scope of set theory. Underlying this approach is the view that by a set-theory one can understand a theory like Zermelo-Fraenkel (with or without *Urelemente*), although this point is generally left implicit (since Suppes works within intuitive set theory).

However, in some situations, it is useful to make the axioms of logic and set theory explicit, particularly if we have grounds for suspecting that they might be questioned. In order to make this point clear, let us recall our division of the axioms of a theory *T* into three levels (see Section 6.2.1), namely: (1) the 'logical' ones (say, classical first-order logic with or without

[915] da Costa's theory will be discussed in the next chapter.

[916] Krause 1990, 1992, 2003a.

[917] At that time, Manin's problem (cf. Chapter 6) was still unknown, as was Dalla Chiara and Toraldo di Francia's quaset theory.

[918] Suppes 1959, Chapter 12; 1967, 2002.

equality), (2) the 'mathematical' ones (say, ZF set theory) and (3) the specific axioms of the theory (for instance, axioms for groups, vector spaces, particle mechanics or quantum mechanics).

As we have already remarked, there are alternatives to this schema, for instance by considering some higher-order logic, perhaps category theory or a set theory other than ZF to fulfil the axioms at (1) and (2). But the important point here is that we may take levels (1) and (2) to be classical first-order logic and ZF. This is in agreement with what has been understood ever since Skolem in the 1920s.[919]

In this schema, any theory *T* is committed to what we have called the Traditional Theory of Identity; that is, to the concept of identity as it is represented within such a logical framework. As we have seen in Chapter 6, despite some differences depending on the level of the language employed (either of first-order or higher-order etc.), the concept of identity is captured by Leibniz's Law, which intuitively asserts that 'two' things are identical if and only if they share all their properties. Hence, classical logic and mathematics in a certain sense vindicate Leibniz's dictum that there cannot be two entities which differ *solo numero*, and this should be so for any theory whose axioms (1) and (2) are 'classical'.

However, as we have seen, the validity of Leibniz's Law and, in particular, of PII in quantum theory has been questioned, and an interesting foundational problem arises, for collections of objects which are indistinguishable conflict with Cantor's 'definition' of the concept of set. Quasi-set theory enters as a way of considering collections of indistinguishable but not identical objects. It is important to note that we are not claiming that quasi-sets are *necessary* in the quantum domain, for there are formulations of quantum mechanics which do not even deal with particles at all.[920] However, *if* our aim is to consider a possible theory which encompasses indistinguishable objects, then it seems that quasi-sets offer an appropriate formal framework.

Generally, as we discussed in Chapter 6, the ways of dealing with indistinguishability within standard mathematics are such that they tend to 'mask' the *distinguishability* of the elements of a set (in the Cantorian sense) by 'passing the quotient' via some equivalence relation and saying that those elements that belong to the same equivalence class are 'indistinguishable'. Alternatively, we could use (permutation) symmetries in such a way that particles of the

[919] Skolem 1922.

[920] Such as, for instance, Ludwig's approach by means of species of structures; see Ludwig 1990.

same kind are treated as 'identical'.[921] But all these approaches, grounded on the idea of invariance under automorphisms, are artificial in the sense that the objects (individuals) are first taken as belonging to a *set* (hence regarded as *distinguishable* entities) and then their individuality is effectively rendered into a 'mock' form by some mathematical device. This kind of trick makes physics work, but the philosophically more interesting problem would be to look for a mathematical framework by means of which certain elements could be taken as indistinguishable in the non-classical sense from the very beginning. This is what quasi-set theory aims to do.

## 7.2 THE QUASI-SET THEORY $\mathcal{Q}$

The form of quasi-set theory we shall present here is a modification of earlier versions.[922] We call such a theory $\mathcal{Q}$, and it is based on ZFU-like axioms (Zermelo-Fraenkel with *Urelemente*). We shall not make explicit the underlying logic of $\mathcal{Q}$, taking it to be subsumed under the general axiomatic scheme of the theory. So, we will suppose that the 'logical' postulates are similar to those of the classical first-order predicate calculus without identity, and we shall not list them here. Only the 'specific' axioms will be mentioned below (that is, those that are specific to quasi-sets).

Furthermore, the talk of 'atoms' and 'quasi-sets' is to be taken as indicating an 'intended interpretation', or *informal semantics* of quasi-set theory, but the commitment to the existence of these entities is not assumed here (as we shall see, we shall touch on the existence of *m*-atoms below). So, we shall proceed as usual in employing an informal language.

Instead of just one kind of atom as in standard ZFU, the theory allows the existence of two sorts of *Urelemente*, termed *m*-atoms and *M*-atoms (two primitive unary predicates express this distinction: m(x) says that $x$ is an *m*-atom and $M(x)$ says that $x$ is an *M*-atom). There are still the binary primitive predicates $\equiv$ (indistinguishability) and $\in$ (membership), one unary functional symbol *qc* (quasi-cardinality) and a unary predicate letter $Z$ ($Z(x)$ says that $x$ is a *set*, and these will correspond to the sets of ZFU). Of course, quasi-set theory can be formulated as a higher-order theory, but this will be not the way we shall present it.

[921] See Castellani 1998a; French and Rickles 2003.

[922] Krause 1996; Dalla Chiara, M. L., Giuntini, R. and Krause, D. 1998.

The basic idea is that the *M*-atoms have the properties of standard *Ur-elemente* of ZFU, while the *m*-atoms may be thought of as representing the elementary basic entities of quantum physics. With regard to these, the concept of identity does not apply. In quasi-set theory, this is achieved by restricting the concept of formula: expressions like $x = y$ are not well formed if $x$ and $y$ denote *m*-atoms. The equality symbol is not a primitive logical symbol, but a concept of *extensional identity* (represented by $=_E$) is introduced by definition so that it has all the properties of standard identity, similar to the corresponding concept in ZFU. Thus, the axiomatics allows us to differentiate between the concepts of (extensional) *identity* (being the very same object) and *indistinguishability* (agreement with respect to all the attributes).

Let us remark at this point that in his book *The Possibility of Metaphysics*,[923] Lowe talks of quasi-objects, a word he uses for "entities which are determinately countable but not always determinately identifiable".[924] These quasi-objects can be exemplified by our *m*-atoms in quasi-set theory, but in our case we take them to be non-individuals. Lowe agrees with the use of quasi-sets when it comes to the possibility of talking of collections of such objects, but wants to preserve the validity of the principle of 'self'-identity (his terms), namely, $\forall x\ (x = x)$ since, as he says, "[t]he property of self-identity is one which, I think, is unproblematically and determinately possessed by quasi-objects such as electrons".[925] However, Lowe does not explain what he means by '(self-)identity' in this context. Although he considers certain identity criteria, there is no reference to a 'definition' of identity, or to a precise characterization of this concept. Generally, identity should be understood in conformity with some kind of logic and the question arises as to which logic he is assuming in his discussion. If it is classical, then Lowe's quasi-objects are semi-classical objects and cannot represent quantum particles in the context of the Received View.

Returning to our theory, a quasi-set (qset for short) $x$ is defined as something which is not an *Urelement*. A qset $x$ may have a cardinal (termed its *quasi cardinal*, and denoted by $qc(x)$), but the idea is that the theory does not associate an ordinal with certain qsets, since there will be quasi sets which cannot be ordered (since their elements are to be indistinguishable *m*-atoms, expressed by the relation $\equiv$). The concept of quasi cardinal is then taken as primitive, since it cannot be defined by the usual means (that is, as particular ordinals). This fits the idea that quantum particles cannot be ordered or

[923] Lowe 1998, Chapter 3, Section 7.

[924] Lowe op. cit., p. 70.

[925] Ibid., p. 71.

counted, but only aggregated in certain amounts. Nevertheless, given the concept of quasi cardinal, there is a sense in saying that there may exist a certain quantity of *m*-atoms obeying certain conditions, although they cannot be named or labelled. (Below we shall also present a variant of the theory $\mathfrak{Q}$ where the quasi-cardinal of a qset $x$ may vary in time—see Section 7.5, as well as a theory $\mathfrak{Q}^m$ which explicitly postulates the existence of *m*-atoms.)

Let us begin by describing the formal details of the theory $\mathfrak{Q}$, by introducing a nominal definition:[926]

**Definition 4.**

(i) **[Quasi-set (qset)]** $Q(x) =_{df} \neg(m(x) \vee M(x))$

(ii) [Pure qset] (*a collection of indistinguishable m-atoms*) $P(x) =_{df} Q(x) \wedge \forall y\,(y \in x \rightarrow m(y)) \wedge \forall y\,\forall z\,(y \in x \wedge z \in x \rightarrow y \equiv z)$

(iii) [Dinge] (*either 'sets' or the* Urelemente) $D(x) =_{df} M(x) \vee Z(x)$ (*these are the '(classical) things', to use Zermelo's original terminology*).[927]

(iv) [A qset whose elements are also qsets] $E(x) =_{df} Q(x) \wedge \forall y\,(y \in x \rightarrow Q(y))$

(v) [Extensional Identity]

$$x =_E y =_{df} (Q(x) \wedge Q(y) \wedge \forall z(z \in x \longleftrightarrow z \in y)) \vee (M(x) \wedge M(y) \wedge \forall_Q z(x \in z \longleftrightarrow y \in z)),$$

*where* $\forall_Q$ *is the universal quantifier relativized to qsets.*

(vi) [Subqset] *For all qsets* $x$ *and* $y$, $x \subseteq y =_{df} \forall z\,(z \in x \rightarrow z \in y)$

The first axioms of $\mathfrak{Q}$ are the following:

(**Q1**) $\forall x(x \equiv x)$

(**Q2**) $\forall x \forall y(x \equiv y \rightarrow y \equiv x)$

(**Q3**) $\forall x \forall y \forall z\,(x \equiv y \wedge y \equiv z \rightarrow x \equiv z)$

(**Q4**) $\forall x \forall y(x =_E y \rightarrow (A(x, x) \rightarrow A(x, y)))$, with the usual syntactic restrictions, that is, $A(x, x)$ is any formula whatever and $A(x, y)$ arises from $A(x, x)$ by the substitution of some free occurrences of $x$ by $y$, provided that $y$ is free for $x$ in $A(x, x)$.

[926] Unless we explicitly state the contrary, all definitions here are nominal, or by abbreviation.

[927] Zermelo 1908.

**Theorem 5.***If either $Q(x)$ or $M(x)$, then $x =_E x$.*

**Proof.** If $Q(x)$, since $\forall z\,(z \in x \leftrightarrow z \in x)$, then $x =_E x$ by the definition of extensional identity. If $M(x)$, then of course for all qset $z$, we have that $x \in z \leftrightarrow x \in z$, so $x =_E x$.▪

**(Q5)** Nothing is at the same time an *m*-atom and an *M*-atom: $\forall x\,(\neg(m(x) \wedge M(x)))$.

**Theorem 6.***If either $Q(x)$ or $M(x)$, then $\neg m(x)$.*

**Proof.** If $Q(x)$, then $\neg m(x)$ by the definition of qset. If $M(x)$, then $\neg m(x)$ by Q5.▪

(**Q6**) The atoms are empty: $\forall x \not\exists y\,(x \in y \rightarrow Q(y))$.

In the following sections we shall discuss the existence of atoms. This last axiom is interesting from the perspective of physics, for it suggests that the *M*-atoms could be 'composed' of *m*-atoms in some way. This is precisely what should be the case, but the relationship between the atoms would not be that of set-theoretical membership. In this case, what seems to be required is some form of mereology suitable for expressing this relationship, but such a theory has yet to be constructed.[928]

(**Q7**) Every set is a qset: $\forall x(Z(x) \rightarrow Q(x))$.

(**Q8**) Qsets whose elements are 'classical things' are sets and conversely:

$$\forall_Q x(\forall y(y \in x \rightarrow D(y)) \leftrightarrow Z(x)).$$

Our intention is to characterize *sets* in $\mathfrak{Q}$ so that they can be identified with the standard sets of ZFU. This could be achieved if they were taken to be those qsets whose transitive closure[929] does not contain *m*-atoms. The '→-part' of Q8 gives half of the answer: if all the elements of $x$ are *Dinge* (sets of *M*-atoms), then $x$ is a set. Concerning the converse, it is not enough to postulate that no element of a set is an *m*-atom, since it may be that the elements of its elements have *m*-atoms as elements and so on. This can be answered if we have $Z(x) \rightarrow \forall y\,(y \in x \rightarrow D(y))$, which is precisely the '←-part' of Q8.

[928] Concerning mereology, see Simons 1987.

[929] This concept has its usual sense: .

(**Q9**) This axiom is the conjunction of the following three sentences:

$$\forall x\,\forall y(m(x) \wedge x \equiv y \longrightarrow m(y))$$
$$\forall x\,\forall y(x =_E y \wedge M(x \longrightarrow M(y))$$
$$\forall x\,\forall y(x =_E y \wedge Z(x) \longrightarrow Z(y))$$

(**Q10**) [The empty qset] There exists a qset (denoted by '∅') which does not have elements: $\exists_Q x\,\forall y(\neg(y \in x))$.

Hence, when using ∅ below, we mean a qset $x$ such that (for instance) $t \in x \leftrightarrow t \in t \wedge t \notin t$. (That it is unique will follow from the whole axiomatics).

**Theorem 7.***The empty qset is a set.*

**Proof.** Take $x =_E \varnothing$ (this expression is an abbreviation, as explained above). Since $y \in x$ is false by Q10, then the antecedent of $\forall y\ (y \in x \rightarrow D(x))$ is true, hence $Z(\varnothing)$ by Q8.▪

(**Q11**) Indistinguishable *Dinge*[930] (see Def. 4) are extensionally identical:

$$\forall_D x\,\forall_D y(x \equiv y \rightarrow x =_E y).$$

**Theorem 8.***The relation of extensional equality has all the properties of classical equality.*

**Proof.** With $x$ such that $D(x)$ then $x \equiv x \rightarrow x =_E x$ by Q11; the axiom Q4 provides substitutivity for *Dinge*; so, the standard axioms for first-order identity are obtained.▪

**Theorem 9.***If $M(x)$ and $x \equiv y$, then $M(y)$; the same holds for 'sets', namely, $Z(x)$ and $x \equiv y$ entails $Z(y)$.*

**Proof.** (For *M*-atoms) Suppose $M(x)$ and $x \equiv y$. If $m(y)$, since $y \equiv x$ by Q2, then we have $m(x)$ by Q9. So, $M(y)$ or $z(y)$. But, by Q11, since $x$ is an

[930] Zermelo's set theory is concerned with a domain (*Bereich*) of individuals, the sets and the *Urelemente*, which he referred to simply as *objects* (*Dinge*).

*M*-atom, $x \equiv y$ entails $x =_E y$, hence by Q4, if $M(x)$ stands for $A(x, x)$, we get $M(y)$. Similar things happen if we suppose $Z(y)$.▪

Remark: The distinction between extensional identity and primitive indistinguishability may be seen as follows, although the formal details can be provided only after other axioms have been stated. By the above axioms and theorems, the indistinguishability relation ≡ permits substitutivity for all primitive non-logical symbols, except membership. That is, if $B$ is $m$, $M$, $Z$ or even $qc$, then $B(x) \wedge x \equiv y \rightarrow B(y)$ is a theorem. If this was possible also for ∈, then since ≡ is reflexive (Axiom Q1), we would have full substitutivity for ≡, hence it could not be distinguished from the usual form of identity.[931] But with regard to membership, this is not the case, that is, $x \in w \wedge y \equiv x$ does not entail that $y \in w$, for the theory has no axioms which entail this fact. So, indistinguishability is not 'standard' identity.

(**Q12**) [Weak-Pair] For all $x$ and $y$, there exists a qset whose elements are indistinguishable from either $x$ or $y$: $\forall x \forall y \exists_Q z \forall t(t \in z \leftrightarrow t \equiv x \vee t \equiv y)$.

We denote this qset by $[x, y]$. When $x$ and $y$ are *Dinge*, we may use the usual notation $\{x, y\}$. Let us remark that $[x, y]$ stands for the qset of elements indistinguishable from either $x$ or $y$, and in general may contain more than two elements.

(**Q13**) [The Separation Schema] By considering the usual syntactical restrictions on the formula $A(t)$, that is, $A(t)$ being a well-formed formula in which $t$ is free, the following is an axiom schema:

$$\forall_Q x \exists_Q y \forall t(t \in y \leftrightarrow t \in x \wedge A(t))$$

This qset is written $[t \in x: A(t)]$, and $\{t \in x: A(t)\}$ when such a qset is a set.

(**Q14**) [Union] $\forall_Q x(E(x) \rightarrow \exists_Q y(\forall z(z \in y) \leftrightarrow \exists t(z \in t \wedge t \in x)))$

This qset is denoted by $\cup x$ or by $\cup_{t \in x} t$ or even by $u \cup v$ when $t$ has just two elements (qsets) $u$ and $v$.

(**Q15**) [Power-qset] $\forall_Q x \exists_Q y \forall t(t \in y \leftrightarrow t \subseteq x)$.

According to the standard notation, we write $\wp(x)$ for this qset.

[931] Shoenfield 1967, Chapter 1.

**Definition 10.**(i)['Ordered pair'] $\langle x, y\rangle =_{df} [\![x], [x, y]\!]$

(ii)[Weak Singleton] $[x] = [x, x]$ (*this is the collection of the indistinguishable from* $x$)

(iii)$x \times y =_{df} [\ \langle z, u\rangle \in \wp\wp(x \cup y)\colon z \in x \wedge u \in y]$

As in the case of $[x, y]$, $[x]$ is the qset of all those elements indistinguishable from $x$, so it may have more than one element. The same may be said for the cartesian product of two qsets etc. The concepts of intersection and difference of qsets are defined in the usual way so that $t \in x \cap y$ iff $t \in x \wedge t \in y$ and $t \in x - y$ iff $t \in x \wedge t \notin y$.

(**Q16**) [Infinity] $\exists_Q x\ (\varnothing \in x \wedge \forall y\ (y \in x \wedge Q(y) \rightarrow y \cup [y] \in x))$.

(**Q17**) [Regularity] (Qsets are well-founded):

$$\forall_Q x(E(x) \wedge x \neq_E \varnothing \rightarrow \exists_Q y(y \in x \wedge y \cap x =_E \varnothing)).$$

Of course this axiom raises another cluster of questions, for if the *m*-atoms are to be thought of as representing elementary particles, then apparently we are faced with the old problem of continuously dividing up a certain object, and our axiom may suggest that we are proposing that such a 'division' will have an end. But of course this is not so, for the axiom talks in terms of qsets; every qset has a *qset* as element with which it has no element in common, but nothing is said about atoms. With regard to these, the problem regarding a suitable mereology remains; in principle we agree with Heisenberg when he said that (in quantum physics) "the concept of 'dividing' has lost its meaning".[932]

## 7.2.1 Relations and Quasi-Functions

In this section we shall see that relations and functions cannot be defined in quasi-set theory as in standard mathematics. The crucial point is that a function cannot distinguish between arguments and values if there are *m*-atoms involved. Furthermore, due to the lack of sense of speaking in $\mathfrak{Q}$ about the identity of and difference between *m*-atoms, ordering relations cannot be adequately defined on a qset that has indistinguishable *m*-atoms as elements.

[932] Heisenberg, 'The nature of elementary particles', reprinted in Castellani 1998.

We shall pay attention to binary relations only, but of course the considerations below can be generalized.

**Definition 11.** *A qset $w$ is a* quasi-relation (*we shall call them simply 'relations') between $x$ and $y$ if it satisfies the following predicate $R$:*

$$R(w) =_{df} Q(w) \wedge \forall z(z \in w \rightarrow \exists u \exists v(u \in x \wedge v \in y \wedge z = E\{u, v\}))$$

As usual we sometimes write $u\ w\ v$ for $\langle u, v \rangle \in w$. Relations are important in general for characterizing the attributes of elements of certain collections of objects. In standard set theory, ordering relations, let us recall, are of two main basic types: *partial* orderings are those binary relations $P$ on a set $A$ which are (i) reflexive, that is, $\forall x(xPx)$, (ii) anti-symmetric, that is, $\forall x \forall y(xPy \wedge yPx \rightarrow x = y)$, and (iii) transitive, $\forall x \forall y \forall z(xPy \wedge yPz \rightarrow xPz)$. If $P$ is also (iv) connected, that is, $\forall x \forall y(x \neq y \rightarrow xPy \vee yPx)$, then it is a *total* (or *linear*) ordering. It is easy to see why such relations cannot be defined on a qset whose elements are indistinguishable *m*-atoms: without the relation of identity, we cannot even state the definition. But what about the so-called *strict* partial and total orderings?

Let us recall that a *strict partial ordering* on a set $A$ is a binary relation $S$ on $A$ such that (i) $S$ is irreflexive, that is, $\forall x \neg(xSx)$ and (ii) transitive. A *strict total ordering* on $A$ is one which is irreflexive, transitive and connected. The reader could rightly say that a binary relation $w$ on a qset $x$ (that is, obeying the predicate $R$ of the above definition) such that (a) $w$ is irreflexive, (b) transitive and (c) for every $u$ and $v$ in $x$ is such that $u\ w\ v \vee v\ w\ u$, should be regarded as a strict total ordering on $x$, although we cannot say that $u$ and $v$ are distinct. What are the consequences of this result?

We also recall that in considering sets, we can always (at least in principle) label any elements, say by associating their singletons with them: for instance, associate $\{x\}$ with $x$. In extensional contexts, this singleton can be viewed as a 'property' of $x$ only (remember that Leibniz's Law holds in classical set theory, so the elements of a set are 'individuals' in a sense).[933] But in $\mathfrak{Q}$ this cannot be done if $x$ is an *m*-atom, for the 'singleton' $[x]$ (as a consequence of the remaining axioms to be stated below) cannot be said to have cardinal 1

[933] More precisely, in the sense according to which all objects of ZF (with the axiom of foundation) can be distinguished from any other since there always exists a set to which one of them belongs while the other one does not—namely, the corresponding singletons.

(in the case considered, we should say 'quasi-cardinal'). So, when we say that $u\ w\ v$, that is, $\langle u, v\rangle \in w$, we should remember that by the definition of the 'ordered pair' given above, $\langle u, v\rangle \in w$ means $[\![u]\!]$, $[u, v]\!]$ *and since* $u \equiv v$, this pair is indistinguishable (in the sense of the axiom of Weak Extensionality to be presented below) from $[\![v]\!]$, $[v, u]\!]$, *which is the 'ordered pair'* $\langle v, u\rangle$. *Furthermore, this qset is also indistinguishable from* $[\![u]\!]$, *that is, from* $\langle u, u\rangle$. *Of course the theory does not imply that* $\langle v, u\rangle$ *(or* $\langle u, u\rangle$*) also belongs to* $w$, but the relation $w$ is indistinguishable (in the same sense) from the relations $w'$ and $w''$, which have these pairs as elements (supposing that the other elements do not provide any further distinction between $w$, $w'$ and $w''$).

All of this means that any strict total order $w$ on a qset $x$ of indistinguishable $m$-atoms such that $\langle u, v\rangle \in w$ is *confused* in the theory with another $w'$ such that $\langle v, u\rangle \in w'$ (the same holds with respect to $w''$ above), and no order could be said to make sense of $x$ within $\Omega$, for the theory cannot distinguish the defined order from another one that has its elements in 'reverse' order (the reader should recall that the idea of a 'reverse order' requires identification of the elements).

This suggests that ordering relations (on a pure qset whose elements are indistinguishable $m$-atoms) don't have detectable significance, for the theory doesn't distinguish between such relations and those which (intuitively speaking) have the 'same' elements in a reverse order. This point has a certain parallel in physics. Suppose that a certain atom releases an electron $u$ becoming an ion. Later, an electron $v$ is captured by the atom, which becomes neutral again. What is the difference between the original atom and the (again) neutral atom? Well, as we know, there are no differences, for quantum physics cannot distinguish between $u$ and $v$ (let us remark that, in our theory, a theorem below will give us a mathematical interpretation of this result). However, there is a sense in saying that $u$ and $v$ are in a certain 'order', for one electron was released while 'another' one was captured. But this 'ordering' is only meta-theoretical (the difference is only in *mente Dei*, as Dalla Chiara and Toraldo di Francia would say). The same can be said concerning the orderings $w$, $w'$ and $w''$ above. Any distinction among them is purely metamathematical (more on this below).

**Definition 12.**[Quasi-functions]*If* $x$ *and* $y$ *are qsets and* $R$ *is the predicate for 'relation' defined above, we say that* $f$ *is a*quasi-function (*q-function*) *with domain* $x$ *and counter-domain* $y$ *if it satisfies the following*

*predicate:*

$$QF(f) =_{df} R(f) \wedge \forall u(u \in x \longrightarrow \exists v(v \in y \wedge \langle u, v \rangle \in f))$$
$$\wedge \forall u \forall u' \forall v \forall v' (\langle u, v \rangle \in f \wedge \langle u', v' \rangle \in f \wedge u \equiv u' \longrightarrow v \equiv v')$$

Furthermore, $f$ is a $q$-injection iff $f$ is a $q$-function from $x$ to $y$ and satisfies the additional condition:

$$\forall u \forall u' \forall v \forall v' (\langle u, v \rangle \in f \wedge \langle u', v' \rangle \in f \wedge v \equiv v' \longrightarrow u \equiv u')$$
$$\wedge qc(Dom(f)) \leq_{\mathbb{E}} qc(Rang(f)).$$

In the same vein, $f$ is a $q$-surjection iff it is a function from $x$ to $y$ such that

$$\forall v(v \in y \rightarrow \exists u(u \in x \wedge \langle u, v \rangle \in f)) \wedge qc(Dom(f)) \geq_{\mathbb{E}} qc(Rang(f)).$$

Finally, an $f$ which is both a $q$-injection and a $q$-surjection is said to be a $q$-bijection. In this case, $qc(Dom(f)) =_E qc(Rang(f))$, where $Dom(F)$ and $Rang(F)$, respectively the domain and the range of $f$, have their usual meanings (but the reader should note that due to the lack of individuality of the $m$-atoms, these qsets have a peculiar characteristic, which is expressed by the Theorem of the Unobservability of Permutations mentioned below). As is easy to see, when there are no $m$-atoms involved, the above concept coincides with the standard definition of a function.

To summarize, we can state the following theorem:

**Theorem 13.** *Neither partial nor total ordering relations can be defined on a pure qset whose elements are indistinguishable from one another.*

**Proof.** (Sketch) The definitions of partial and total orders require antisymmetry, and this property cannot be stated without identity. Asymmetry also cannot be supposed. In fact, if $x \equiv y$, then for every $R$ such that $\langle x, y \rangle \in R$, it follows that $\langle x, y \rangle =_E [\![x]\!] =_E \langle y, x \rangle \in R$; so, $xRy$ entails $yRx$.▪

## 7.2.2 Quasi-Cardinals

In order to present the remaining axioms, we need to show that a 'copy' of ZFU can be defined within $\mathfrak{Q}$. Let us consider the main idea. First, we need to define a translation from the language of ZFU to the language of $\mathfrak{Q}$. This will show that the theory $\mathfrak{Q}$ encompasses a 'classical' counterpart which coincides with ZFU.

The translation can be defined in the following way: let $A$ be a formula of the language of ZFU (which we may admit has a unary predicate $S$ which stands for 'sets'). Then, call $A^q$ its translation to the language of $\mathfrak{Q}$, defined as follows:

(i) If $A$ is $S(x)$, then $A^q$ is $Z(x)$
(ii) If $A$ is $x = y$, then $A^q$ is $((M(x) \wedge M(y)) \vee (Z(y) \wedge Z(y)) \wedge x =_E y)$
(iii) If $A$ is $x \in y$, then $A^q$ is $((M(x) \vee Z(x)) \wedge Z(y)) \wedge x \in y$
(iv) If $A$ is $\neg B$, then $A^q$ is $\neg B^q$
(v) If $A$ is $B \vee C$, then $A^q$ is $B^q \vee C^q$
(vi) If $A$ is $\forall x\, B$, then $A^q$ is $\forall x\, (M(x) \vee Z(x) \rightarrow B)$

**Theorem 14.** *If $A$ is an axiom of* ZFU *and $A^q$ is its translation into the language of $\mathfrak{Q}$ given by the above definition, then $A^q$ is a theorem of $\mathfrak{Q}$.*

This theorem, whose proof can be given by careful checking, shows that if $\mathfrak{Q}$ is consistent, so is ZFU.

The above result shows that there is a *copy* of ZFU in $\mathfrak{Q}$ (Figure 7.1). In this 'copy', we may define as usual the following concepts: $Cd(x)$ for '$x$ is a cardinal'; $card(x)$ denotes 'the cardinal of $x$, and $Fin(x)$ says that' $x$ is a finite quasi-set'. Then, by considering these concepts, we may present the axioms for quasi-cardinals:

**(Q18)** Every object which is not a qset (that is, every *Urelement*) has quasi-cardinal zero: $\forall x\, (\neg Q(x) \rightarrow qc(x) =_E 0)$.

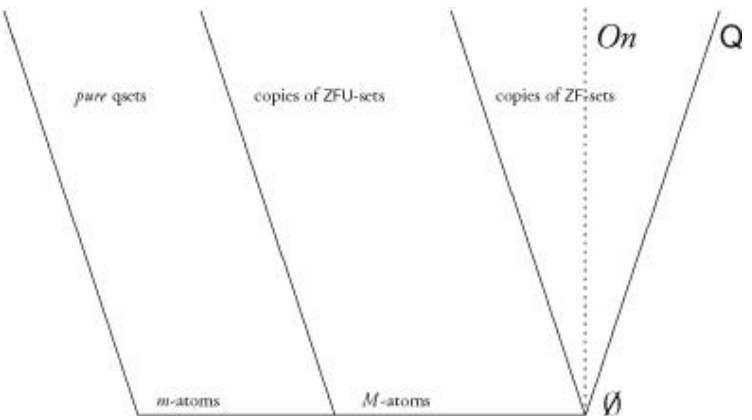


Figure 7.1. The Quasi-Set Universe.

**(Q19)** The quasi-cardinal of a qset is a cardinal (defined in the 'classical part' of the theory) and coincides with its cardinal itself when this qset is a set:[934]

$$\forall_{Q}x\,\exists!y(Cd(y)\wedge y=_{\mathbb{E}Q}c(x)\wedge(Z(x)\rightarrow y=_{\mathbb{E}}card(x)))$$

So, we are postulating that any qset has a quasi-cardinal and that such a quasi-cardinal is a cardinal (as defined by the usual means in the 'standard' part of the theory). This axiom may appear to be contrary to the result mentioned in the preceding section which shows that no order relation can be defined on a qset of indistinguishable *m*-atoms, for the existence of its quasi-cardinal seems to suggest that any quasi-set (including those whose elements are indistinguishable *m*-atoms) *can* be ordered, for if defined as usual, a cardinal is a particular ordinal. The explanation of this apparent anomaly is that the associated ordinal of a quasi-set of indistinguishable *m*-atoms cannot be something that belongs to the 'classical' part of the theory (in the figure above, it would lie in the region above *m* of the 'pure' qsets). Alternatively, we could say, as in the case of Skolem's paradox, that such an ordinal does not belong to the theory at all (in the sense that its existence cannot be derived from the axioms). This is of course an interesting point to be further considered, but we shall not do so here.

However, the consequence is that perhaps for physical applications, a different concept of cardinal should be used instead of the standard one, for at least in those cases involving quantum objects it would be useful to have a process of counting which does not induce any kind of order. For instance, perhaps we could take a definition in the sense of Frege-Russell (in terms of a certain class of equinumerous classes).[935] Another alternative would be to use Enderton's definition of the concept of kcard:[936] Enderton suggests that kcard(x) is to be understood as the set of all sets *y* equinumerous to *x* and having the least possible rank. Then, *kcard*(*x*) = *kcard*(*y*) iff *x* and *y* are equinumerous, as desired. However, the definition of kcard relies on regularity, and not on the axiom of choice (which in the standard definition is used to show that every set has a cardinal number),[937] so for a finite non-empty set *x*, *kcard*(*x*) fails to be a natural number (which perhaps is not so convenient for physics). Perhaps we could eliminate from the axiom above the fact that

[934] Actually, 'sets in $\mathfrak{Q}$', that is, 'copies' of ZFU sets.

[935] More precisely, the cardinal of a set *X* would be $card(X) =_{df} \{Y: Y \approx X\}$, where ≈ means 'equinumerous to' (Suppes 1957, p. 109). But the problems with such a definition are well known (op. cit.).

[936] We thank Antonio Coelho for reminding us of this definition.

[937] Enderton 1977, p. 222.

*qc*(*x*) is a cardinal and leave its characterization to the particular model to be considered. Whichever option is chosen, it seems clear that the search for a more adequate definition of cardinal (and of course of 'counting') as far as quantum physics is concerned is something to be further investigated.

Finally, we remark that in the context of quantum field theory, certain situations might arise for which we could not say that the quantity of elements in a certain qset is fixed (the case of virtual particles comes to mind).[938] But it is not necessary to decide this issue here, for the possibility of varying the quasi-cardinal in time (as given in the theory $\mathfrak{Q}'$ sketched below) and the concept of *the cloud* of a qset (also given below) will provide the mathematical insights for the relevant discussion. Let us turn now to the other axioms.

**(Q20)** Every non-empty qset has a non-null quasi-cardinal:

$$\forall_Q x(x \neq_E \varnothing \rightarrow qc(x) \neq_E 0)$$

**(Q21)** $\forall_Q x\ (qc(x) =_E \alpha \rightarrow \forall \beta\ (\beta \leq_E \alpha \rightarrow \exists_Q y\ (y \subseteq x \wedge qc(y) =_E \beta)))$
**(Q22)** $\forall_Q x\ \forall_Q y\ (y \subseteq x \rightarrow qc(y) \leq_E qc(x))$
**(Q23)** $\forall_Q x\ \forall_Q y\ (Fin(x) \wedge x \subset y \rightarrow qc(x) <_E qc(y))$
**(Q24)** $\forall_Q x\ \forall_Q y\ (\forall w\ (w \notin x \vee w \notin y) \rightarrow qc(x \cup y) =_E qc(x) + qc(y))$In the next axiom, $2^{qc(x)}$ denotes (intuitively) the quantity of subquasi-sets of *x*. Then,
**(Q25)** $\forall_Q x\ (qc(\mathfrak{p}(x)) =_E 2^{qc(x)})$

If the concept of identity is inapplicable for *m*-atoms, how can we ensure that a qset *x* such that $qc(x) =_E \alpha$ has precisely $2^\alpha$ subqsets? In standard set theories (as in the 'classical part' of $\mathfrak{Q}$, that is, in considering those qsets which fit the sets of ZFU), as is well known, if *card*(*x*) denotes the cardinal of *x*, then by the definition of exponentiation of cardinals, $2^{card(x)}$ is defined to be the cardinal of the set ${}^x2$, which is the set of all functions from *x* to the Boolean algebra 2 = {0, 1}.[939] In $\mathfrak{Q}$ this definition doesn't work. Let us explain why.

Suppose that α is the quasi-cardinal of *x*, which is a cardinal by Q19. This axiom says that every qset has a unique quasi-cardinal which is a cardinal (defined in the 'classical part' of the theory), and if the qset is in particular a set (in $\mathfrak{Q}$), then this quasi-cardinal is its cardinal *stricto sensu*. So, every quasi-cardinal is a cardinal and the above expression 'there is a unique …' makes sense. Furthermore, from the fact that ∅ is a set, it follows that its

[938] Cf. Chapter 9; Toraldo di Francia 1985.

[939] Enderton 1977, p. 141.

quasi-cardinal is 0. Then we may define

$$2^{qc(x)} =_{\mathrm{df}} qc({}^{\alpha}2) \tag{7.2.1}$$

and then, since α is a cardinal and both α and 2 are *sets* in $\mathfrak{Q}$ (that is, they are copies of ZFU-sets), we have

$$2^{qc(x)} =_{\mathfrak{E}} card({}^{\alpha}2). \tag{7.2.2}$$

So, we may take the cardinal of the qset ${}^{\alpha}2$ in its usual sense to mean $2^{qc(x)}$. Then, this last equality gives meaning to the axiom Q25, since it explains what $2^{qc(x)}$ means: it is the cardinal of the *set* of all the applications from α (the quasi-cardinal of $x$) in 2. By considering this, the axiom may be written as follows, where $x$ is a qset and α is its quasi-cardinal:

$$\textbf{Axiom Q25}(\text{AlternativeForm})\ \forall_{Q} x(qc(\mathcal{P}(x)) =_{\mathfrak{E}} card({}^{\alpha}2)).$$

We remark that the second member of this equality has a precise meaning in $\mathfrak{Q}$, since both α and 2 act as in classical set theories, as remarked above, since they are (copies of) *sets*. This characterization allows us to avoid a further problem: we recall that in standard set theories we can prove that $\mathfrak{p}(x)$ is equinumerous with ${}^{x}2$ by defining a one-one function $f$: $\mathfrak{p}(x) \to 2$ as follows: for every $y \subseteq x$, let $f(y)$ be the characteristic function of $y$, namely, the function $\chi_y$: $x \to 2$ defined by

$$\chi_y(t) =_{\mathrm{df}} \begin{cases} 1 & \text{if } t \in y \\ 0 & \text{if } t \in x - y \end{cases} \tag{7.2.3}$$

Then any function $h \in {}^{x}2$ belongs to the range of $f$ since

$$h = f(\{t \in x : h(t) = 1\}).$$

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Suppose now that $x$ is a qset such that $qc(x)$ is the natural number $n$ and that all elements of $x$ are indistinguishable from each other (the natural numbers are defined in $\mathfrak{Q}$ in the usual way, just as in the model of ZFU we have defined in $\mathfrak{Q}$).[941] In this case, we cannot define the characteristic quasi-function $\chi_y^q$ for $y \subseteq x$, since, for instance, if $\chi_y^q(t) =_{\mathfrak{E}} 1$ for $t \in y$, then $\chi_y^q(t) =_{\mathfrak{E}} 1$ as

[940] Enderton op. cit., p. 131.

[941] For our purposes, it is sufficient to consider finite qsets (this definition is also standard; roughly speaking, their quasi-cardinals are natural numbers).

well for every $w \in x$, independently of whether $w$ belongs to $y$ or not. This is due to the definition of the quasi-functions given above, since for every quasi-function $f$,

$$\langle a,b\rangle \in f \wedge \langle c,d\rangle \in f \wedge a \equiv c \rightarrow b \equiv d.$$

In other words, if the image of a certain $t$ by the quasi-function $f$ is 1, then the image of every element indistinguishable from $t$ will be 1 as well. So, $\mathfrak{Q}$ distinguishes only between *two* quasi-functions from $x$ to 2, namely, that which associates 1 with all elements of $x$ and that which associates 0 with all of them. This is the reason why we have used $qc({}^{\alpha}2)$ to mean $2^{qc(x)}$, since both $\alpha$ and 2 may be viewed as *sets* (in the standard sense). If we had used ${}^{x}2$ instead, we would be unable to distinguish among certain quasi-functions, so complicating the meaning of Q25, since we could have no way of counting the number of subquasi-sets of a qset. But, by using ${}^{\alpha}2$, since both $\alpha$ and 2 behave 'classically', Q25 retains its usual meaning.

From these considerations, we may conclude that when $x$ is a qset whose elements are indistinguishable $m$-atoms, we cannot prove *within* $\mathfrak{Q}$ that $qc(x) =_E n$, and so we cannot *count* $2^n$ subquasi-sets in $x$. Since this is precisely what Q25 intuitively says, we may affirm that this axiom cannot be proven from the remaining axioms of $\mathfrak{Q}$. But, since it holds for particular qsets, namely, those which are $\mathfrak{Q}$-copies of *sets*, it cannot be disproved either. In order to show that Q25 cannot be disproved, consider the *sets* in $\mathfrak{Q}$; since they behave as classical sets, we can prove that what Q25 asserts is true. Now it suffices to take a qset whose elements are indistinguishable $m$-atoms and such that $qc(x) =_E \alpha$.

Axiom Q25 has another important implication for $\mathfrak{Q}$. In standard set theories, if $card(x)$ is (say) the natural number $n$, then there are exactly $n$ subsets of $x$ which are singletons. Can this result be proved also in $\mathfrak{Q}$? If not, how can we make sense of the idea that if $qc(x) =_E n$, then $x$ has $n$ elements? We recall once more that the main motivation for $\mathfrak{Q}$ is the way quantum mechanics deals with elementary particles and then, although there is a sense in saying that, say, there are $k$ electrons in a certain level of a certain atom, there is no way of counting them or of distinguishing them, as we have already said (see also below). These considerations motivate the definitions and axioms of the next section.

## 7.2.3 'Weak' Extensionality

If $x$ is a qset whose elements are indistinguishable from one another as above (let us suppose again that $qc(x) =_E n$, which suffices for our purposes), then the singletons $y \subseteq x$ are indistinguishable from each other, as follows from the Weak Extensionality axiom Q26 below. So, all the singletons (in the intuitive sense) seem to fall into just one qset. But it should be recalled that these 'singletons' (subqsets whose quasi-cardinality is 1) are not *identical* (that is, they cannot be proven to be *the same object* in the theory), although they are indistinguishable in a precise sense (given by Q26 below). In other words, although the theory cannot distinguish between them, we cannot affirm either that they are the same qsets or that their elements are identical. So, it is consistent with $\mathfrak{Q}$ to suppose that if $qc(x) =_E \alpha$, then $x$ has precisely $\alpha$ 'singletons' (which are of course not of the form $[y]$ given above). So, due to Q25, the theory does not forbid the existence of such singletons, despite the fact that in $\mathfrak{Q}$ we cannot prove that they exist as 'distinct' entities, and hence we may reason in $\mathfrak{Q}$ as physicists do when dealing with a certain number of indistinguishable quanta or with collections of them.

The absence of a theory of identity for the *m*-atoms requires a modification of the usual Axiom of Extensionality of standard set theories,[942] which here does not hold. In order to do so, let us introduce the following definition:

**Definition 15.***For all non empty quasi-sets $x$ and $y$,*

(i) $Sim(x, y) =_{df} \forall z \forall t\ (z \in x \wedge t \in y \rightarrow z \equiv t)$. In this case we say that $x$ and $y$ are similar.
(ii) $QS\ im(x, y) =_{df} Sim(x, y) \wedge qc(x) =_E qc(y)$. That is, $x$ and $y$ are $Q$-similar iff they are similar and have the same quasi-cardinality.

In the axiom below, $x/\equiv$ stands for the quotient qset of some qset $x$ by the equivalence relation $\equiv$.

**(Q26)** [Weak Extensionality] Qsets which have the same quantity of elements of the same sort are indistinguishable. In symbols,

[942] As is well known, this axiom says that those sets which have the *same* elements are identical.

$$\forall_Q x \forall_Q y((\forall z(z \in x / \equiv \longrightarrow \exists t(t \in y / \equiv \wedge \wedge QS\ im(zt))))$$
$$\wedge \forall t(t \in y / \equiv \longrightarrow \exists z(z \in x / \equiv \wedge \wedge QS\ im(t,z)))$$

It is easy to see that if there are no *m*-atoms involved, so that ≡ becomes the usual identity, then the axiom coincides with the standard axiom of extensionality used in ZFC.

As a consequence, it is easy to prove the following theorem:

**Theorem 16.**

(i) $x =_E \varnothing \wedge y =_E \varnothing \rightarrow x \equiv y$
(ii) $\forall_Q x \forall_Q y\ (S\ im(x, y) \wedge qc(x) =_E qc(y) \rightarrow x \equiv y)$
(iii) $\forall_Q x \forall_Q y\ (\forall\ z\ (z \in x \leftrightarrow z \in y) \rightarrow x \equiv y)$
(iv) $x \equiv y \wedge qc([x]) =_E qc([y]) \leftrightarrow [x] \equiv [y]$

One of the main applications of the Weak Extensionality axiom is the theorem of the Unobservability of Permutations to be presented below, which provides a way of representing within quasi-set theory the idea that if a certain object is 'permuted' with an indistinguishable one, then 'nothing changes at all'! Of course this has no meaning in standard mathematics, due to the lack of sense in speaking of indistinguishable but not 'identical' objects, as we have seen elsewhere. Furthermore, the standard axiom of extensionality would then apply, and hence any permutation of non-identical objects would give us a *different* set. In what follows we shall show other applications of this axiom, but before that let us comment on the replacement axioms of $\mathfrak{Q}$.

## 7.2.4 Replacement Axioms

We may add to quasi-set theory certain replacement axioms as follows. If $A(x, y)$ is a formula in which $x$ and $y$ are free variables, we say that $A(x, y)$ defines a $y$-(*quasi-functional*) condition on the quasi-set $t$ if $\forall w\ (w \in t \rightarrow \exists s\ A(w, s) \wedge \forall w \forall w'\ (w \in t \wedge w' \in t \rightarrow \forall s \forall s'\ (A(w, s) \wedge A(w', s') \wedge w \equiv w' \rightarrow s \equiv s'))$ (this is abbreviated by $\forall x\ \exists\ !\ yA(x,y)$). Then, we have:

**(Q27)** [Replacement Axioms]

$$\forall x \exists! y A(x,y) \rightarrow \forall_Q u \exists_Q v(\forall z(z \in v \rightarrow \exists w(w \in u \wedge A(w,z))).$$

Intuitively speaking, this axiom says that the images of qsets by quasi-functions are also qsets. It is easy to see that when there are no *m*-atoms involved, that is, when $\mathfrak{Q}$ becomes essentially ZFU, then this schema coincides with the standard replacement axioms. The difference here is the way of stating the schema, for we must obey the restriction imposed above on the concept of quasi-function.

## 7.2.5 The Strong Singleton

An important concept in quasi-set theory is that of the strong singleton of an element $x$ (either a qset or an atom). This is a qset with quasi-cardinality 1 whose 'only element' is indistinguishable from $x$. It is interesting that, contrary to what would be expected, we cannot prove that this element *is* $x$. Thus, we will find ourselves in a situation according to which we will be able to say that we have *just one* element of a certain kind but without the theoretical means of identifying it, even in principle. Let us look at the details.

**Definition 17.** *A* strong singleton *of $x$ is a quasi-set $x'$ which satisfies the following property:*

$$x' \subseteq [x] \wedge qc(x') =_E 1.$$

In other words, a strong singleton of $x$, as remarked above, is a qset $x'$ whose only element is indistinguishable from $x$. In standard set theories, this qset is of course the singleton *stricto sensu* whose only element is $x$ itself, but here $x$ may be an *m*-atom, and in this case there is no way of speaking of something *being identical to* $x$. Even so, we can prove that such a qset exists:

**Theorem 18.** *For all $x$, there exists a strong singleton of $x$.*

**Proof.** The qset $[x]$ exists by the weak pair axiom. Since $x \in [x]$ (since $\equiv$ is reflexive), we have that $qc([x]) \geq_E 1$ by Q20. But, by Q21, there exists a subqset of $[x]$ which has quasi-cardinal 1. Take this qset to be $x'$. ▪

**Theorem 19.** *All the strong singletons of $x$ are indistinguishable.*

**Proof.** Immediate consequence of Q26, since all of them have the same quasi-cardinality 1 and their elements are indistinguishable by definition.▪

It is important to note that, as we shall see, we cannot prove that the strong singletons of $x$ are extensionally identical. With regard to indistinguishable *m*-atoms, we cannot give ostensive definitions, say by putting our finger over an *m*-atom and saying 'This is Peter'. Even so, as in quantum physics, we may reason as if a certain element does or does not belong to the qset; the law of the Excluded Middle $x \in y \vee x \notin y$ remains valid, even if we cannot verify which case holds. This idea fits with what happens with the electrons in an atom; in general we know how many electrons there are, and we can say that some of them *are* in that atom, but we cannot tell *which* particular electrons are in the atom: the identity of the electrons has been 'lost', but we prefer to say that there is no identity to be lost.

**Theorem 20.** *For all qsets $x$ and $y$, if $y \subseteq x$ and $x$ is finite, then $qc(x - y) =_E qc(x) - qc(y)$.*

**Proof** By definition, $t \in x - y$ iff $t \in x \wedge t \notin y$. Then $(x - y) \cap y =_E \varnothing$. Hence, by Q25, $qc((x - y) \cup y) =_E qc(x - y) + qc(y)$ (let us call this expression (i)). But, since $y \subseteq x$, $(x - y) \cup y =_E x$ and so, in order for (i) to be true, $qc(x - y) =_E qc(x) - qc(y)$. ▪

The next result may be regarded as a quasi-set version of the Indistinguishability Postulate, which, we recall, says that permutations of indistinguishable quanta are not observable. In order to state and prove this result, we introduce a definition.[943]

**Definition 21.**

(i) *Let $x$ be a qset such that $E(x)$, that is (according to Definition 1), its elements are also qsets. Then,*

$$\cap x =_{df} (z \in \cup x : \forall s(s \in x \rightarrow z \in s))$$

(ii) If $m(u)$,[944] *then* $S_u =_{df} [s \in \wp([u]) : u \in s]$

(iii) $u^* =_{df} \cap_{t \in S_u} t$

[943] The Lemmas that follow were originally formulated by A. Sartorelli.

[944] The case where $u$ is an *m*-atom is of course the most interesting one. The generalization of this definition to *M*-atoms and sets is immediate, but in this case the result coincides with the analogous (and sometimes trivial) situation in standard set theories. The case of *m*-atoms is really the one which makes the difference.

**Lemma 22.***If m(u), then:*

(i) $u \in \cup_{t \in S_u} t$
(ii) $\forall s\ (s \in S_u \rightarrow u \in s)$
(iii) $z \in u^*$ iff $z \in \cup_{t \in S_u} t \wedge \forall_s (s \in S_u \rightarrow z \in s)$
(iv) $u \in u^*$
(v) $u^* \subseteq [u]$
(vi) *If* $s \in S_u$, *then* $u^* \subseteq s$

**Proof.** (i) $z \in \cup_{t \in S_u} t$ iff $\exists t\ (t \in S_u \wedge z \in t)$. Therefore, by the above definition, $z \in \cup_{t \in S_u} t$ iff $\exists t\ (t \in \mathfrak{p}([u]) \wedge u \in t \wedge z \in t)$. But since $[u] \in \mathfrak{p}([u])$ and $u \in [u]$, it follows that $u \in \cup_{t \in S_u} t$. (ii) $\forall s\ (s \in S_u \leftrightarrow s \in \mathfrak{p}([u]) \wedge u \in s)$. Therefore, $\forall s\ (s \in S_u \rightarrow u \in s)$. (iii) is an immediate consequence of the above definition. (iv) is an immediate consequence of (i)–(iii). (v) Suppose that $z \in u^*$. By (iii), we have $\forall s\ (s \in S_u \rightarrow z \in s)$. But since $[u] \in S_u$, it follows that $z \in [u]$. (vi) If $z \in u^*$, then, as before, $\forall s\ (s \in S_u \rightarrow z \in s)$. But, by hypothesis, $s \in S_u$; so, $z \in s$. ▪

**Lemma 23.***If u is an m-atom and z is a qset, then if* $z \subseteq u^*$ *and* $qc(z) =_E 1$, *it follows that* $u \in u^* - z$ or $qc(u^*) =_E 1$.

**Proof.** Suppose that $u \notin u^* - z$. Since $u \in u^*$, it follows that $y \in z$. But $z \subseteq u^* \subseteq [u]$, therefore $z \in S_u$. But, by the above Lemma (6), $u^* \subseteq z$. By hypothesis, $z \subseteq u^*$, hence $u^* =_E z$, and so $qc(u^*) =_E qc(z) =_E 1$. ▪

**Theorem 24.***For every u,* $qc(u^*) =_E 1$.

**Proof.** By (iv) of Lemma (22), $u^* \neq_E 0$. So, by Q20, $qc(u^*) \neq_E 0$, hence $qc(u^*) \geq_E 1$. We shall show that the equality holds. Suppose that $qc(u^*) >_E 1$. Then, by Q21, there exists a qset $w \subseteq u^*$ such that $qc(w) =_E 1$. So, by Lemma (23), $u \in u^* - w$. But $u^* - w \subseteq [u]$, since $u^* \subseteq [u]$, therefore, by (v) of Lemma (22), $u^* \subseteq u^* - w$. However, since $u^* - w \subseteq u^*$, it follows that $u^* =_E u^* - w$. Again by Q20, $w \neq_E \varnothing$ since $qc(w) =_E 1$. Then let $t \in w$. So, $t \in u^*$ since $w \subseteq u^*$, hence $t \in u^* - w$ (since $u^* =_E u^* - w$). Thus $t \notin w$, a contradiction. ▪

**Lemma 25.***For all m-atoms u and v, if* $u \equiv v$, *then* $u^* \equiv v^*$. *Furthermore, if* $u \in w$, *then* $u^* \subseteq w$ *for any qset w.*

**Proof.** If $u \equiv v$, then $u^* \subseteq [u]$ and $v^* \subseteq [v]$, so $S\,im(u^*, v^*)$ (see Definition (15)). But, by Theorem (24), $qc(u^*) =_E 1$ and $qc(v^*) =_E 1$ and then, by Lemma (23), $u^* \equiv v^*$. The last part can be proven by noting that if $u \in w$, then $u \in w \cap [u]$, so as $w \cap [u] \subseteq [u]$, therefore $w \cap [u] \subseteq S_u$. Then, by (v) of Lemma (22), $u^* \subseteq w \cap [u]$ and so $u^* \subseteq w$. ▪

These last results show that $u^*$ is, as expected, a strong singleton of $u$. The remarkable fact is that we cannot prove that $u^* \equiv v^*$ entails $u^* =_E v^*$. This is due to the fact that nothing in the theory can ensure that *thatm*-atom which belongs to $u^*$ *is the samem*-atom which belongs to $v^*$, since neither the expression $u = v$ nor $u =_E v$ are well formed in this case. Furthermore, it is worth recalling that the usual Extensionality Axiom, which could be used for expressing this fact, is not an axiom of our theory but, instead, we have the 'weak' axiom Q26 which refers to indistinguishability only, and not identity. The impossibility of proving the above result should not be regarded as a deficiency of the theory, but rather as expressing the fact that it is closer to what happens in quantum physics than standard set theories. The next theorem reinforces this point.

## 7.2.6 Permutations are not Observable

If we think of material bodies as collections of quanta of some sort, as intuitively we may think of that wall in front of us as 'composed' of atoms, protons, electrons and the like, a first attempt at approaching a mathematical characterization is to regard it as a *set* endowed with some kind of structure. For instance, Noll's definition of a continuum body says that such a body is a triple $\langle B, \Phi, m \rangle$ *where B* is an arbitrary *set*, $\Phi$ is a set of mappings (hence, also *sets*) from *B* into *E*, the three-dimensional Euclidean point space, and *m* is a function defined on the subsets of *B* into the set $\mathfrak{R}$ of real numbers; the set *B* is *the set of particles* of the continuum body. There follow some axioms which provide the desired 'structure', but which do not interest us here.[945]

Granted that the analogous definition in quantum theory would be more complicated, a 'quantum body' should still be regarded as a *collection* of some sort, plus something which expresses the 'structural' characteristics (for, as Toraldo di Francia noted, a simple collection of objects does not fit our concept of a physical object: as he says, "put together a million billiard balls and try

[945] See Ignatieff 1996, Chapter 9.

to see if we can observe something interesting").[946] Of course, it is well known from our knowledge of isomers that collections of 'identical' atoms may yield completely different substances, as for instance $C_2H_6O$ may stand for both $CH_3-CH_2-OH$, ethyl alcohol and $H_3C-O-CH_3$), dimethyl ether. But, setting such issues aside, the Indistinguishability Postulate (**IP**) holds and hence, quoting Penrose again,

> according to the modern theory [QM], if a particle of a person's body were exchanged with a similar particle in one of the bricks of this house then nothing would have happened at all.[947]

The next theorem is the quasi-set version of this principle.

**Theorem 26.**[Unobservability of Permutations]*Let $x$ be a finite qset such that $x \neq_E [z]$ and $z$ an m-atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists $w'$ such that*

$$(x - z') \cup w' \equiv x$$

**Proof.** Case 1: $t \in z'$ does not belong to $x$. In this case, $x - z' =_E x$ and so we may admit the existence of $w'$ such that its unique element $s$ belongs to $x$ (for instance, $s$ may be $z$ itself); then $(x - z') \cup w' =_E x$. Case 2: $t \in z'$ does belong to $x$. Then $qc(x - z') =_E qc(x) - 1$ by the above Theorem (20). We then take $w'$ such that its element is $w$ itself, and so it follows that $(x - z') \cap w' =_E \varnothing$. Hence, by Q25, $qc((x - z') \cup w') =_E qc(x)$. This intuitively says that both $(x - z') \cup w'$ and $x$ have the same quantity of indistinguishable elements. So, by applying Q27 (see above), we obtain the result. ▪

Supposing that $x$ has $n$ elements, then if we 'exchange' elements $z$ by the corresponding indistinguishable elements $w$ (set theoretically, this means performing the operation $x - z' \cup w'$), then the resulting qset remains indistinguishable from the original one. In a certain sense, it is not important, from a pragmatic point of view, if it is either $x$ or $x - z' \cup w'$ that we are dealing with.

[946] Toraldo di Francia 1990, p. 17.

[947] Penrose 1989, p. 360.

## 7.2.7 The Axiom of Choice

Finally, the theory $\mathfrak{Q}$ can be supplied with a version of an 'axiom of choice':

**(Q28)** [The Axiom of Choice]

$$\forall_Q x(E(x) \wedge \forall y \forall z(y \in x \wedge z \in x \rightarrow y \cap z =_E \varnothing \wedge y \neq_E \varnothing) \rightarrow \exists_Q u \forall y \forall v(y \in x \wedge v \in y \rightarrow \exists_Q w(w \subseteq [v] \wedge qc(w) \wedge w \cap y \equiv w \cap u)))$$

Intuitively speaking, if $x$ is a qset whose elements are disjointed non-empty qsets, then there exists a qset $u$ such that for every $y \in x$ and $v \in y$, $u$ has as element one that is indistinguishable from $v$ (which is expressed by the last part of the axiom). In other words, we may form a qset which has elements indistinguishable from the members of the elements of $x$. Using Russell's well-known example of the pairs of socks, we are not collecting in the 'choice qset' $u$ one element of each pair of socks, but a sock indistinguishable from each element of the pairs of socks (which may, of course, be one of them, but we can never prove that). Standard mathematics masks this discussion due to extensionality, but maybe quasi-set theory can provide the tools for a new perspective.

Of course the discussion could be further explored along several lines. For instance, since the idea of a strong singleton of $x$ gives a qset with just *one* element indistinguishable from $x$ (due to its quasi-cardinality), then apparently we could think of using such a qset in any sentence involving *one*$x$. In other words, the existence of such strong singletons could act in the theory as Hilbert's ε-symbol, which, as is well known, enables us to prove as a theorem the sentence which expresses the axiom of choice in standard set theories (like ZF). So, perhaps some form of the axiom of choice would necessarily result from any theory involving indistinguishability. But this is a matter for future analysis. Anyway, such responses would require that collections of *m*-atoms could be constructed in the theory, and the above axioms do not postulate the existence of *Urelemente*. We shall comment on this point in the next section.

## 7.2.8 Remark on the Existence of Atoms: The Theory $\mathfrak{Q}^m$

The careful reader has surely noted that in $\mathfrak{Q}$ we have not postulated the existence of atoms. This is in accordance with most theories involving *Urelemente*.

But for the purposes of physics perhaps it would be interesting to postulate the existence of such entities (and also collections of them). If it were necessary for particular applications, we could add to our theory an axiom saying explicitly that (say) qsets of *m*-atoms exist. Let us call $\mathfrak{Q}^m$ such a variant of the above theory. Thus, its specific axiom would be

$$[\text{Axiom of}\mathfrak{Q}^m]\ \exists_Q x(x \neq \varnothing \wedge \forall y(y \in x \rightarrow m(y)))$$

As a result, a model of quasi-set theory may no longer be obtained within ZFC (in other words, we guess that $\mathfrak{Q}^m$ is not interpretable in ZFU in the sense presented in the next section). This bears upon the mathematical analysis of the peculiarities of such 'new' quasi-sets. However, this is a topic to be further developed elsewhere.[948]

# 7.3 RELATIVE CONSISTENCY

Let us return to the formal theory $\mathfrak{Q}$. In the preceding sections we have seen that there is a 'copy' of ZFU in $\mathfrak{Q}$. This proves that if $\mathfrak{Q}$ is consistent, so is ZFU (and, hence, so is ZF) (see Fig. **7.1**); now let us see what happens when we consider the reverse conditional. We shall see that there is a sense in saying that we can 'mimic' the behaviour of quasi sets in ZF (hence in ZFU).[949] This of course does not show that all we can do in $\mathfrak{Q}$ can also be done in ZF (or in ZFU), as we shall see below, for obvious reasons, unless we keep the discussion inside a certain structure. In other words, the most we can say is that *a* and *b* are indistinguishable relative to a structure $\mathfrak{A}$ (for instance, *a* and *b* are indistinguishable in $\mathfrak{A}$ if and only if they are invariant under automorphisms of $\mathfrak{A}$).[950] However, even in this case *a* and *b* are objects of the universe of ZF; hence they are *individuals* in the sense of obeying the classical theory of identity, although their distinction can't be seen from the point of view of the structure. In quasi-set theory, the existence of objects which are indistinguishable from the point of view of the whole theory, that is, from

[948] Tzouvaras has recently made some interesting remarks on this point in an e-mail to the authors.

[949] It is left open whether the same can be said concerning the theory $\mathfrak{Q}^m$. We suggest that this theory cannot be embedded, in the sense given below, in ZFC.

[950] See Krause and Coelho, 2005. Roughly speaking, inside the structure some elements may be invariant under automorphisms, and so are 'indistinguishable' from the point of view of the structure according to the standard definitions (ibid.). But even so they can be distinguished *from the outside* since in the Zermelo-Fraenkel universe they are always distinguishable; for instance, add to the structure the singletons of these elements, which suffices to make the distinction.

the point of view of the whole model of $\mathfrak{Q}$ is allowed, contrary to classical mathematics.

In this section, we will work within ZFC. Let $m$ be a non-empty set and let $R$ be an equivalence relation on $m$. The equivalence classes of the quotient set $m/R$ are denoted by $C_1$, $C_2$ …. If $x \in m$, define $\hat{x} = \{x, C_x\}$, where $C_x$ is the equivalence class to which $x$ belongs and call $\hat{m}$ the set of all $\hat{x}$ with $x \in m$.

Let $X$ be the set $X = \hat{m} \cup M$, where $\hat{m}$ is as above and $M$ is a set such that $\hat{m} \cap M = \varnothing$ and $rank\hat{m} = rank M$.

Then we define a superstructure **Q** over the set $X$, called the *Q-set universe* (see Fig. **7.2**). As we will see, **Q** acts as a 'model' for the quasi-set theory $\mathfrak{Q}$. The definition is as follows:

$\mathcal{Q}_0 =_{df} X$

$\mathcal{Q}_1 =_{df} X \cup \wp(X)$

□

$\mathcal{Q}_\lambda =_{df} \cup_{\beta<\lambda} \mathcal{Q}_\beta$ if λ is a limit ordinal $\mathcal{Q} =_{df} \cup_{\alpha \in On} \mathcal{Q}_\alpha$

In accordance with the terminology of $\mathfrak{Q}$, the elements of $M$ are called $M$-atoms, $M$-elements or $M$-objects, while the elements of $\hat{m}$ are called $m$-atoms, $m$-elements or $m$-objects. The final goal is to interpret the basic elements of $\mathfrak{Q}$ in the corresponding objects in **Q**.

For the sake of simplicity, we introduce another superstructure which we will call $\mathbf{Q}^s$, constructed in a similar way to **Q** above but having only the set

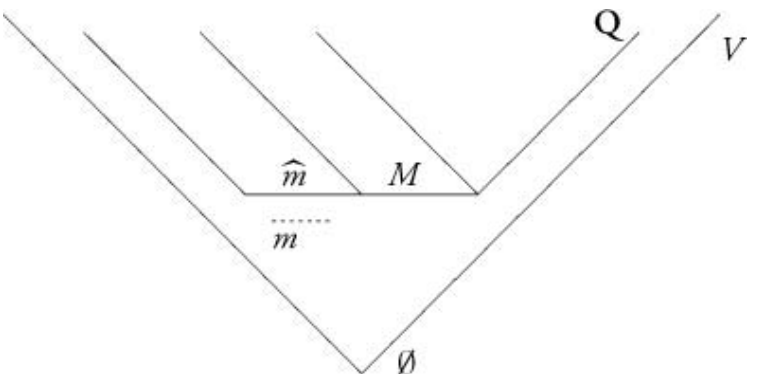


Figure 7.2. Simulating qsets in $V$ (the ZFC-universe). The elements of $m$—dashed lines—are outside **Q**.

*M* as its 'ground' basis instead of the whole *X*. The sets of **Q** (that is, those $x$ that satisfy the predicate $Z(x)$) will be the only elements of $\mathbf{Q}^s$.

Now we define a translation from the language of $\mathfrak{Q}$ into the language of ZFC. But first let us define on the quotient set $\hat{m}$ the following relation, which is an equivalence relation, as is easy to see:

$$\hat{x} \sim \hat{y} \text{ iff } C_x = C_y.$$

(7.3.1)

If $\hat{x} \sim \hat{y}$, we say that $x$ and $y$ are *indistinguishable*. We note that in this way we are identifying $x$ and $y$ by means of the class (or 'state', or 'sort') they are in, represented by the equivalence classes they belong to, and this is done without direct reference to the objects themselves, which lie *outside* of the structure **Q**.[951] This approach essentially corresponds to Weyl's idea of 'aggregates' of individuals.

Let us now turn to the translation. Suppose that $A$ is an atomic formula of the language of $\mathfrak{Q}$; let us call $A'$ its translation into the language of ZFC.[952] We suppose that all the sets (of ZFC) involved in the definition below belong to **Q** and that the quantifiers are restricted to this class. Then, the translation says:

(i) If $A$ is $m(x)$, then $A'$ is $x \in \hat{m}$.
(ii) If $A$ is $M(x)$, then $A'$ is $x \in M$
(iii) If $A$ is $Z(x)$, then $A'$ is $x \in \mathcal{D}^s \wedge x \notin M$
(iv) The translation of the term $ac(x)$, is $card(x)$, the cardinal of the set $x$.
(v) If $A$ is $x \equiv y$, then $A'$ is $\left(\left(x \in \hat{m} \wedge y \in \hat{m} \wedge x \sim y\right) \vee x = y\right)$
(vi) If $A$ is $x \in y$, then $A'$ is $x \in y$

The other formulas are translated in the usual way. By means of the above procedure, the definitions of $\mathfrak{Q}$ can be translated into ZFC. Let us give some examples:

1. In $\mathfrak{Q}$, a quasi-set is an object which is neither an *m*-atom nor an *M*-atom. The formal definition is $Q(x) =_{df} \neg(m(x) \vee M(x))$, as we have seen. Due to the translation, in ZFC this simply means that $x \in \mathbf{Q}$ but neither $x \in \hat{m}$. That is, a set, which in ZFC 'represents' a quasi-set, is a set of the class **Q** that neither belongs to *M* nor is an ordered pair of the form $\langle x, C_x \rangle$.
2. In $\mathfrak{Q}$, the 'pure' quasi-sets are those quasi-sets whose elements are only *m*-atoms. In the present case, they are interpreted (in ZFC) as subsets

[951] Recall that the rank of the elements of  is lower than the rank of , and hence of the elements of *X*. See Fig. 7.2.

[952] We use $x, y, \ldots$ as individual variables in both theories.

of $\hat{m}$. Furthermore, in $\mathfrak{Q}$ we define a *classical object* as an $x$ which obeys the predicate $C$ defined by $C(x) =_{df} M(x) \vee Z(x)$. This simply means that $x$ is either an element of $M$ or of $\mathbf{Q}^s$.

3. Quasi-set inclusion is defined as in ZFC;[953] thus, its translation coincides with the standard inclusion of sets in such a theory.
4. Extensional Equality expresses (in ZFC) the usual identity governed by the axiom of extensionality (in the case of sets) or the identity of the elements of the set $M$.

Now we turn to a detailed translation of the axioms of $\mathfrak{Q}$ so that the translated formulas can be proven to be theorems of ZFC. Those axioms of $\mathfrak{Q}$ which are adaptations of the axioms of ZFU have obvious translations.[954] We remark that, in $\mathfrak{Q}$, the Axioms of Indistinguishability state that $\equiv$ has the properties of an equivalence relation. If we consider the above translation, it is easy to see that the images of the pairs $\langle x, y \rangle$ *such that* $x \equiv y$ define an equivalence relation in ZFC. Concerning extensional identity, since this means nothing more than the usual identity for certain sets of ZFC, it follows that the substitutivity law is also valid. So, the translations of the axioms (Q1) throughout (Q4) are true in ZFC.

The other axioms of $\mathfrak{Q}$ and their translations can be dealt with according to the above definitions. Then, by careful checking, we can see that all these translations are true in the defined model.[955]

What is the meaning of this result? We have constructed in ZFC a model for quasi-set theory. Hence, if ZFC is consistent, so is $\mathfrak{Q}$. Does this result entail that quasi-set theory is not necessary and that everything can be done within classical mathematics, thus apparently contradicting what we have said above? Such a suggestion would be completely misleading, and a misinterpretation of all we have done. Let us explain what is going on.

An important and distinctive feature of the 'model' **Q** is that the rank of the sets $x$ and $y$ is smaller than the rank of the elements of the set $X\left(X = \hat{m} \cup M\right)$ *which is the ground set of the structure*.[956] In other words, $x$ and $y$ (being elements of the set $m$) are *outside* the model **Q**; hence, we cannot talk about either their identity or their diversity *within* **Q**, thus respecting the basic idea involving quasi-sets. But of course we can do this *from the outside*, for

[953] That is, $x \subseteq y$ iff $\forall z\,(z \in x \rightarrow z \in y)$.

[954] In the present case, the elements of $X$ play the rôle of the *Urelemente* in **Q**: the elements of act as the $m$-atoms while those of $M$ act as the atoms in ZFU.

[955] An example of this work for an early version of the quasi-set theory is given in da Costa and Krause 1999.

[956] The *rank* of a set $A$ is the least ordinal $\alpha$ such that $A \subseteq V_\alpha$.

instance in the well-founded model $\mathscr{V} = \langle V, \in \rangle$ of ZFC, in the same vein as we can *talk* about the identity and about the diversity of two electrons in a natural language (say, in English), although this does not make much sense in quantum mechanics, as we have seen. However, when we look *inside* **Q** and ask for its *internal logic*, then of course it cannot be classical logic. This kind of logic we call *non-reflexive*, meaning a logic in which the 'traditional' (Leibnizian) theory of identity of classical logic does not hold in full. The meaning of this expression 'internal logic' can be seen exactly in the same sense as when one asks for the internal logic of a *topos* in category theory. There, as is well known, it is said that such a logic is intuitionistic logic, since the partially ordered sets of 'sub-objects' of a given 'object'[957] is not a Boolean algebra, but a Heyting algebra.[958] What happens with collections of indistinguishable things, like quantum objects, is in a certain sense analogous to what happens with topoi. But a remark made by Hatcher can be applied here; paraphrasing, we may say that we can study quasi-sets (as collections of non-individual quanta) from the set-theoretical point of view, as the physicist does when he uses classical mathematics. However, this is the *external* logic, which can be chosen to be, as Hatcher says, "anything we choose it to be";[959] but the *internal* logic of quasi-set theory (hence, of the structure **Q**) is a kind of non-reflexive logic, in the sense defined in the next chapter. Furthermore, if we consider the theory $\mathfrak{Q}^m$ introduced above, which postulates the existence of qsets of *m*-atoms, the above translation doesn't work, and we speculate that no translation from (the languages of) $\mathfrak{Q}$ to ZFC can be defined at all. So, if this is the case, $\mathfrak{Q}^m$ *is not* equivalent to ZFC. The search for such a translation is still an open problem.

Anyhow, even keeping within $\mathfrak{Q}$, if someone wants to dismiss it by saying that owing to the above result it doesn't differ from ZFC, we recall that intuitionistic logic also has an interpretation in modal logic S4, but it can surely not be claimed that intuitionistic logic has no intrinsic merit owing to this fact. We shall reveal other kinds of advantages associated with $\mathfrak{Q}$ in the next sections. Furthermore, finding a model for $\mathfrak{Q}$ does not take us to the core of the philosophical problem of finding an adequate language to express indistinguishability right from the start, for in the whole ZF model $\mathscr{V} = \langle V, \in \rangle$, every object (set) can be individuated in the sense that its singleton can always be formed. Thus, this alternative looks much

[957] This terminology came from category theory.

[958] See Hatcher 1982, p. 303.

[959] op. cit., p. 303.

like the usual procedure of 'masking' the *ab ovo* individuality attributed to the basic objects and considering only their role within specific sub-structures where the individuality is not 'seen'. As we have emphasized, the interesting philosophical project is just to work with indistinguishable objects from the start.

# 7.4 QUASET IDEAS WITHIN QUASI-SET THEORY

As we saw in Chapter 5, there exists a kind of 'epistemic' indeterminacy of an element relative to a quaset, in the sense that given an object $x$ and a quaset $y$ it is not always possible to know whether $x$ belongs to $y$ or not. This can be mapped into the framework of quasi-set theory as follows.

First of all, let us recall that in Dalla Chiara and Toraldo di Francia's quaset theory there are two basic membership relations: $x \notin y$ is read '$x$ certainly does not belong to $y$' and $x \in y$ is read '$x$ certainly belongs to $y$'. It is important to note that $x \notin y$ is not the same as $\neg(x \in y)$. Here, to avoid difficulties with the terminology, we shall introduce the following definitions, given in [Ω].

**Definition 27.**

$$x \varepsilon y =_{df} \exists z(z \in y \wedge z \equiv x)$$

Thus, its negation $x \not\varepsilon y$ reads $\forall z\,(x \in y \rightarrow z \equiv x)$. In this case, following the terminology of quaset theory, we say that *$x$certainly does not belong to$y$*, and that it certainly belongs to $y$ otherwise. Intuitively, we may say that when $z\,\varepsilon\,y$, there are 'traces' of $x$ in $y$, for there are elements indistinguishable from $x$ in $y$. By using the above concept, we can introduce something like the dual of the quasi-extension of a quaset; previously, the quasi-extension of a quaset $x$ was defined as the collection of all objects that certainly belong to $x$. Now, we can introduce what we can call *the fuzzy complement* of $x$ relative to a certain previously given quasi-set $z$ (such that $x \subseteq z$) to indicate the collection (quasi-set) of the elements of $z$ for which we cannot affirm that they certainly do not belong to $x$. If we call $Cl_z(x)$ such a quasi-set (the *cloud* of $x$ relative to $z$), then:

**Definition 28 [The Cloud of$x$relative to$z$]**

$$Cl_z(x) =_{df} [t \in z : t \varepsilon x]$$

The elements of $z$ for which we can't be sure that they certainly do not belong to $x$ may be said to be 'elements in potentia' of $x$, as in Dalla Chiara and Toraldo di Francia's quaset theory, although they do not use this terminology. This qset $z$ may also be said to be *the environment* of $x$, that is, the 'place' from where $x$ can exchange its elements. We remark that since $x$ may be a subqset of distinct $z$'s, by making the variable $z$ range over a certain collection of qsets, the quasi-cardinal of the cloud of $x$ remains undetermined.[960] Furthermore, we can define a quasi-function $a^{\dagger}$: $z \mapsto z$ as follows: $a^{\dagger}$ associates with $w$ a qset whose elements are indistinguishable from the elements of $x$ and whose quasi-cardinal is $qc(w) + 1$, which can be done by considering $w$ the cloud of some $x$ for suitable $z$'s. In the same way, we may define a quasi-function $a$ which intuitively decreases the quasi-cardinal of $x$ by one unit, among other possibilities. It is possible that these quasi-functions could act as the creation and the annihilation operators in quantum field theory.

We note that at this stage, quasi-set theory does not provide ways of dealing with 'interactions' among $m$-atoms, as might be useful when considering relativistic aspects of quantum theory. But we think that it is possible to extend the theory to encompass such operations, by conveniently defining, for instance, how the union of two strong singletons of $x$ and $y$ (being $m$-atoms which have certain specified properties) can yield another qset with $m$-atoms of a different sort. The use of sortal predicates may be useful here.

Continuing with our discussion, it is easy to prove, for instance, the following results:

**Theorem 29.***If $x \subseteq z$, then:*

(a) $\forall_Q x \forall_Q z (Cl_z(x) = \bigcup_{t \in x} [t] \cap z)$
(b) $Cl_z(\varnothing) =_E \varnothing$ and $Cl_{\varnothing}(x) =_E \varnothing$
(c) *For every* $z$, $Z(x) \rightarrow Cl_z(x) =_E x$

The concept of the cloud of a qset suggests the idea that a qset $x$ is something *in between* its extension $Ext(x)$, namely, the qset of the objects that certainly belong to $x$ and its cloud $Cl_z(x)$ for some $z$. Due to the fact that we can't identify the elements of a pure qset, we may say that a qset is not strictly determined by its elements, hence, some degree of intensionality is also present here, as noted by Dalla Chiara and Toraldo di Francia with regard

[960] This idea may be useful in discussing quantum field theories.

to their quasets. But since quasi-sets are collections of objects (sometimes only *m*-atoms) which either do or do not belong to them, they can also be taken as 'extensional entities' of a sort, and this marks one difference between quasi-sets and quasets (see more on this below). So, quasi-set theory enables us to interpret the collections from both perspectives: an intensional one and an extensional one.

The concept of the cloud of $x$ may also be useful for understanding some basic facts about quasi-sets, since it gives us an interpretation of what it means to say that a certain non-individual belongs to a certain collection of entities. Think for instance of a finite quasi-set $x$ whose elements (by simplicity) are indistinguishable *m*-atoms. Then the axioms of the theory $\mathfrak{Q}$ tell us that there is a quasi-cardinal which stands for the number of elements of $x$. But since these elements don't have individuality, how can we verify if a certain object, say $t$, either belongs to $x$ or not? First of all, we should remark that $t$, as used here, cannot be understood as a *proper name* of an *m*-atom, for we can't reason in quasi-set theory according to our standard ('classical') practice. It is one thing to discuss the concepts in the metalanguage, which is 'classical', and where we can talk about 'this' or 'that' *m*-atom, but in order to be precise, we need to look at the right formulas written in the language of $\mathfrak{Q}$. In the present case, to say that (a certain) $t$ does belong to $x$ is to say that there is something like $t$ in $x$ which behaves as we expect $t$ to behave: in our terminology, we say there is an element indistinguishable from $t$ in $x$, that is, $t \, \mathbf{\varepsilon} \, x$.

The above theorem about the Unobservability of Permutations also helps in fixing this interpretation. Let us recall that this theorem says intuitively that if we exchange an element of $x$ with an indistinguishable one, then the resulting qset is indistinguishable from the original one. The axioms of $\mathfrak{Q}$ state that the membership relation $\in$ behaves as in standard set theories, contrary to what happens in the theory of quasets, but owing to the non-individuality of the *m*-atoms, we cannot have, say, a decision procedure (even for finite sets) for checking whether a certain element belongs to a qset or not. The axioms act always in the conditional form: *if* a certain object belongs to a certain qset, then this or that. The concept of the cloud of a qset expresses the idea of a collection of those elements of which it is false to say that they certainly do not belong to the considered quaset.

Since the *m*-atoms do not have individuality, then there is still a certain epistemic indeterminacy whether a certain element belongs to a certain qset or not; all we can say is that there may be traces of something which behaves like such an element in the collection, but we can never say that we are

talking about *that* element. We always have elements (this is particularly important for non-individual entities) which 'could be' in $x$, but we cannot affirm that they really are or are not elements of $x$, since there may be elements indistinguishable from them in $x$. In quasi-set theory, as in quaset theory, the quasi-cardinal of a quasi-set is well determined, but now, since the quasi-set $x$ may be a subset of some $z$, we can't affirm the same about the quasi-cardinality of the cloud of $x$, except if we fix the qset $z$. But, if in the expression $Cl_z(x)$ the variable $z$ remains free, then the quasi-cardinal of the collection remains undetermined.

Of course, the idea of the cloud of $x$ may suggest several applications, but in what follows we shall explore the idea that the quasi-cardinal of a quasi-set may not be well defined. Perhaps there is a link here with the idea of virtual particles and with quantum field theory, as anticipated by Toraldo di Francia,[961] although he has spoken in terms of quasets. In quantum field theories even the quantity of elements in a certain state might be indeterminate owing to the creation/annihilation processes, so in certain situations even the cardinal of the collections might be not defined (see below). To give a short account of such a situation, let us modify the theory $\mathfrak{Q}$ by exchanging the axioms for quasi-cardinals in order to give meaning to the claim that the quasi-cardinal of a quasi-set may vary in time. This is the topic of the next section.

## 7.5 CHANGES IN TIME: THE THEORY $\mathfrak{Q}^t$

The intended interpretation of the $m$-atoms of our theory as elementary particles suggests the idea that the qsets could stand for certain 'states' in which these particles might be. Of course this is a problematic issue, for we need to explain the meaning of the term 'state', and this is not so easy to do, even in quantum theory. But even without a detailed discussion, some aspects of this idea might be captured by our formalism and in describing this point we will have the opportunity to mention some of the philosophical and mathematical difficulties one faces in describing in 'set-theoretical' terms collections of particles when other assumptions are considered, such as, for instance, the existence of virtual particles.

[961] Toraldo di Francia 1978.

As we have already noted in Chapter 5, Dalla Chiara and Toraldo di Francia have said that the cardinality of a quaset may vary in time, and of course the same might be said for the quasi-cardinal of a quasi-set. Let us consider this possibility by describing how it is possible to map 'the changes in time' of Weyl's *aggregates*[962] into the scope of quasi-set theory.

As we have noted, Weyl considered the mathematical treatment of finite collections of objects so that, intuitively speaking, each one of the elements may be in a certain 'state' and the only information we may have concerns the number of elements of the whole collection (the *aggregate*) that there are in each state, but such that there is no possibility of identifying the elements that belong to each particular state. Then, as we have emphasized, the fundamental idea concerning Weyl's 'effective aggregates' cannot be accurately described within the scope of the classical theory of sets since a set is a collection of *distinguishable* objects and, so, it is not possible to maintain the idea that the elements that belong to a certain state should not be (even in principle) distinguished from one another. This can be seen by recalling once more that any structure built in ZFC can be embedded in a 'rigid' structure (that in which any object is an individual).[963] Intuitively, this means that although we can consider some 'objects' as indistinguishable, for instance by means of automorphism invariance, the structure where this is considered can always be embedded in a more general one (within $\langle V, \in \rangle$) where all the objects are distinguishable; as already remarked, it suffices to add to the structure the singletons of the considered elements.

In our theory, the effective aggregates in Weyl's sense may be considered as qsets $x/\equiv$ (the quotient of $x$ by the indistinguishability relation), where $x$ is a pure qset, that is, a qset containing only *m*-atoms as elements. That is to say, the *states* can be viewed as the equivalence classes of elements of $x$ by the indistinguishability relation; since the full concept of identity lacks sense for the *m*-atoms, only their quantity in each 'state' (that is, the quasi-cardinality of each equivalence class) may be known, and this conforms to Weyl's intuitive idea.

But in doing so we are considering such aggregates only 'statically', that is, without consideration of the possibility that the quantity of elements in a state may vary in time. However, Weyl himself considered this

[962] Weyl 1949, App. B; see also Krause 1991.

[963] See the previous chapter; Krause and Coelho 2005.

possibility:

> As long as elements are capable of discrete states only, we are forced to dissolve time into a succession of discrete moments,
>
> $$t = \ldots, -2, -1, 0, 1, 2, \ldots.$$
>
> Transition of the system from its state at the time $t$ into its state at time $t + 1$ will then be a jump-like mutation. With the $n$ elements individualized by the labels $p$, the changing state of affairs will be described by giving the state $C(x; t)$ of the element $p$ at the time $t$ as a function of $p$ and $t$. This 'individual' description, by means of the function $C(x; t)$, is to be supplemented by the principle of relativity according to which the association between the individuals and their identification marks $p$ is a matter of arbitrary choice; but it is an association for all time, and once established it is not to be tampered with. If, on the other hand, at each moment attention is given to the visible states only, then the numbers $n_1(t)$, …, $n_k(t)$ in their dependence on $t$ contain the complete picture—however incomplete this information is from the 'individualistic' standpoint. For now we are told only how many elements, namely $n_i(t)$, are found in the state $C_i$ at any time $t$, but no clues are available whereby to follow up the identity of the $n$ individuals through time; we do not know, nor is it proper to ask, whether an element that is now in the state, say $C_5$, was a moment before in the state $C_2$ or $C_6$. The world is created, as it were, anew at every moment, no bond of identity joins the beings present at this moment with those encountered in the next. This is a philosophical attitude towards the changing world taken by the early Islamic philosophers, the Mutakallimûn. This non-individualizing description is applicable even if the total number $n_1(t) + \cdots + n_k(t) = n(t)$ of elements does not remain constant in time.[964]

It is possible to map these ideas into the scope of our formalism if we conveniently modify the theory $\mathfrak{Q}$. Let us call $\mathfrak{Q}^t$ the theory obtained by modifying the language of $\mathfrak{Q}$ as follows. The new language encompasses:

(i) two additional binary predicate symbols: $<$ and $=_t$;
(ii) a binary functional symbol: *tcard*;
(iii) a ternary predicate symbol: $\varepsilon$;
(iv) a denumerably infinite collection of variables of 'second species': $t'$, $t''$, ….

The individual variables of the language of $\mathfrak{Q}$ will be called variables of first species and the functional symbol qc does not appear in this new language.

[964] Weyl op. cit., pp. 242–3.

We use $x, y, \ldots$ and $t, t_1, \ldots$ as syntactical variables ranging over the collections of variables of first and of second sort respectively. The variables of second kind are called 'time variables'.

Among the terms, we have now the following additional ones: if $x$ and $t$ are variables of first and of second species respectively, then *tcard*$(x, t)$ is also a term.

The set of atomic formulas is increased by the following, where $x$ and $y$ are variables of first species and $t_1$ and $t_2$ are variables of second species: $\varepsilon(x, y, t)$, $t_1 < t_2$ and $t_1 =_t t_2$.

The formulas in general are defined by the usual procedures, by observing that we are now dealing with a two-sorted language. The new terms and atomic formulas may be intuitively thought of in the following sense: $t_1 < t_2$ says that time $t_1$ precedes time $t_2$ in the order $<$; $\varepsilon(x, y, t)$ will be written $x \in^t y$ and means that $x$ is an element of $y$ at the time $t$. Finally, the term tcard(x,t) stands for the quasi-cardinal of the quasi-set $x$ at the time $t$.

The postulates of $\mathfrak{Q}^t$ are essentially those of $\mathfrak{Q}$ (alternatively, we could use $\mathfrak{Q}^m$ instead), but instead of the axioms for qc, we have similar axioms for tcard (see below), plus the following ones (we recall that by the definition of atomic formulas, $<$ and $=_t$ must be flanked by time variables only):

**(T1)** $<$ is irreflexive, anti-symmetric and transitive
**(T2)** $=_t$ is reflexive and obeys the substitutivity principle

These axioms impose the usual interpretation on the time relations, and time may be thought of as a succession of discrete moments. (T2) says that $=_t$ is like classical identity for instants of time. Then, a time variable $t$ may be thought of as ranging over a totally ordered set of instants of time. Obviously, alternative approaches are possible; for instance, instead of a totally ordered set (which may be discrete), we could postulate that the set of instants of time is a complete ordered field, such as for instance the set of reals. In this way, we could talk about an 'interval' of time in the usual intuitive sense, such as for instance a closed interval of the real number line. For such an alternative, it is sufficient to change the axioms (T1) and (T2) above for the axioms for a complete ordered field, plus an obvious adaptation of the language.

The axioms for the concept of *tcard* are essentially those we used for qc, with appropriate adaptations for the case we are considering here; that is, the primitive concept of cardinal is now a binary functional symbol whose first and second arguments must be occupied respectively by variables of first kind and by time variables. We recall that *card*$(x)$ stands for the cardinal of the

set $x$, defined in the 'classical part' of the theory, $\exists!$ means 'there is exactly one', $Cd(y)$ stands for '$y$ is a cardinal' (also defined in the qset copy of ZFU), $[x]$ denotes the qset of the objects which are indistinguishable from $x$ (which in general has more than one element) and $\mathfrak{p}(x)$ is the power-qset of $x$. The axioms are the following:

**(Q18t)** $\forall_Q x\, \forall t\, (\neg Q(x) \rightarrow tcard(x, t) =_E 0)$
**(Q19t)** $\forall_Q x\, \forall t\, \exists !\, y\, (Cd(y) \wedge y =_E tcard(x, t) \wedge (Z(x) \rightarrow y =_E card(x)))$
**(Q20t)** $\forall_Q x\, \forall t\, (x \neq_E \varnothing \rightarrow tcard(x, t) \neq_E 0)$
**(Q21t)** $\forall_Q x\, \forall t\, (tcard(x, t) =_E \alpha \rightarrow \forall \beta\, (\beta \leq \alpha \rightarrow \exists_Q y\, (y \subseteq x \wedge tcard(y, t) =_E \beta))$
**(Q22t)** $\forall_Q x\, \forall_Q y\, \forall t\, (y \subseteq x \rightarrow tcard(x, t) \leq tcard(x, t))$
**(Q23t)** $\forall_Q x\, \forall_Q y\, \forall t\, (Fin(x) \wedge x \subset y \rightarrow tcard(x, t) < tcard(y,t))$
**(Q24t)** $\forall_Q x\, \forall_Q y\, \forall t\, (\forall\, w\, (w \notin x \vee w \notin y) \rightarrow tcard(x \cup y, t) =_E tcard(x, t) + tcard(y, t))$
**(Q25t)** $\forall_Q x\, \forall t\, (tcard(\mathfrak{p}\, (x), t) =_E 2^{tcard(x,t)})$

The above axioms, as is easy to see, preserve all the desired properties of the quasi-cardinal of a qset (which, by force of Q19t, is also a cardinal) in each instant of time. Then, Weyl's aggregates are qsets $x/\equiv$, where $x$ is a pure qset and $\equiv$ is our indistinguishability relation, but now the quasi-cardinal of the equivalence classes may vary in time.

A link with our previous discussion can be given as follows. Suppose that the qset $z$ in the definition of the cloud of $x$ relative to $z$ may vary, that is, $z$ ranges over the sequence of qsets $z' \subseteq z'' \subseteq z'''$ …. We can associate this sequence with a time sequence $t$, $t'$, $t''$, …, so that tcard($Cl^i_z(x)$, $t$) may vary. Then, we have a precise sense of saying that the quasi-cardinal of some qset whose elements are varying in number varies in time. The details of course can be developed straightforwardly.

## 7.6 QUANTUM STATISTICS WITHIN $\mathfrak{Q}$

As we have seen earlier, the derivation of quantum statistics depends on the assumption of the Indistinguishability Postulate (IP). In this section we will see that in using the language of quasi-set theory, we may avoid such a hypothesis. This of course comes at a price, namely, the corresponding complication in the language, since we need to work within the theory $\mathfrak{Q}$. But the gain with

respect to other features is interesting, for we can see, for instance, that in considering (truly) indistinguishable objects, we arrive 'naturally' at quantum statistics, which reinforces the idea that IP is imposed by the necessity of discourse, that is, its existence is due to the difficulty we have in dealing with *ab ovo* indistinguishable objects within 'standard' languages such as those of classical logic and mathematics.[965] Let us provide a sketch of the main ideas involved in the derivation of the quantum distribution functions within $\mathfrak{Q}$. We begin by showing how to obtain the electronic distribution of an atom; for instance, in taking a sodium atom, we shall give a meaning, in terms of quasi-sets, to the expression $1s^2\ 2s^2\ 2p^6\ 3s^1$. Restricted quantifiers shall be used: so $\forall_A x\alpha$ means $\forall x\ (x \in A \rightarrow \alpha)$, while $\exists_A x\alpha$ stands for $\exists x\ (x \in A \wedge \alpha)$.

Suppose that $\mathfrak{p}$ is a collection of non-empty pure qsets, that is, their elements are *m*-atoms. The elements of $\cup\mathcal{P}$ will be called *quantum objects*.[966] Let $S$ be a finite and totally ordered 'set' (in the sense of $\mathfrak{Q}$ of *quantum states*, whose elements are denoted by $s_1, \ldots, s_n$. Furthermore, let $F$ be a unary predicate defined as follows: $F(x) =_{df} m(x) \wedge \forall_S s\ (x \in s \rightarrow qc(s) =_E 1)$. In words, $F(x)$ says that $x$ is a quantum object which cannot belong to a state with more than one element; such an $x$ shall be called a *fermion*; also, let us define $B(x) =_{df} m(x) \wedge \neg F(x)$ and call such an $x$ a *boson*.[967]

Now let $R$ be the quasi-relation

$$R =_{df} [[p,s]: p \in \mathcal{P} \wedge s \in S], \tag{7.6.1}$$

which is subjected to the following restriction: for every $[p, s] \in R$, if $x \in p$ obeys $F(x)$, then $qc(p) =_E 1$. That is, there can be no more than one fermion in each quantum state. Call this restriction *Pauli's Principle* (PP).

For instance, let us consider our example of the electrons of a sodium atom. So, $qc(\cup\mathcal{P}) =_E 11$ (there are eleven electrons to be distributed), and let $s =_E \{s_1, \ldots, s_{12}\}$ be the set of possible states (the reason we are taking twelve states will be made clear below). So, in this case the particular relation $R$ given by equation (7.6.1) is

$$R =_{df} [[p_1, s_1], \ldots, [p_{12}, s_{12}]], \tag{7.6.2}$$

[965] This section is based on Krause, Sant'Anna and Volkov 1999 and on Sant'Anna and Santos 2000.

[966] We remark that such an hypothesis has a physical motivation. For instance, we could be concerned with certain collections of particles, such as those listed in a particle table. Furthermore, the elements of the collections in $\cup\mathcal{P}$ are not necessarily indistinguishable.

[967] So, if $\mathfrak{Q}$ is consistent, then no $x$ can be at the same time a fermion and a boson. The given definitions ensure that a given particle is either a fermion or a boson, but we could modify them conveniently in order to admit within our description other kinds of particles as well, hence other kinds of statistics, such as those mentioned previously (parastatistics, say). But in this section we shall just consider bosons and fermions to exemplify the technique of using quasi-sets.

where $qc(p_i) =_E 1$ ($i =_E 1, \ldots, 11$), and $qc(p_{12}) =_E 0$.

In the general case, with the relation $R$ defined as above (equation (7.6.1), we can select a family $\{S_i\}_{i \in I}$ ($I =_E \{1, \ldots, n\}$) of subsets of $S$ such that $\cap S_i =_{\mathbb{E}} \varnothing$ . For instance, in our example, we may have

$$\{S_i\}_{i \in \{1,\ldots,12\}} =_{\mathrm{df}} \{ \{s_1, s_2\}, \{s_3, s_4\}, \{s_5, s_6, s_7, s_8, s_9, s_{10}\}, \{s_{11}, s_{12}\} \}. \tag{7.6.3}$$

Each $S_i$ may be called an *energy bin* (or *macrostate*), while each element $s \in S_i$ may be called an *energy state* (or *microstate*). The mention of a particular $S_i$ (which is generally done via physical criteria) particularizes the collections of particles to certain interesting states, namely, the states that belong to the selected set $S_i$.

Taking our example once again, we see that each element of $\{S_i\}_{i \in \{1, \ldots, 12\}}$ (equation 7.6.3) corresponds to an energy level in the sodium atom. So, the element

$$\{s_5, s_6, s_7, s_8, s_9, s_{10}\} \in \{S_i\}_{i \in \{1,\ldots,12\}}$$

corresponds to the energy level $2p$, which encompasses six electrons, since each $s_i$, $i = 5, \ldots, 10$ is associated with a quasi-set with just one quantum object (such a qset, remember, is a *strong singleton*). The element $\{s_{11}, s_{12}\}$ corresponds to the energy level $3s$, which allows two electrons but in this case 'has' just one electron (recall that $qc(s_{11}) =_E 1$ and $qc(s_{12}) =_E 0$). The other two elements of $\{S_i\}_{i \in \{1, \ldots, 12\}}$, namely, $\{s_1, s_2\}$ and $\{s_3, s_4\}$, correspond respectively to the energy levels $1s$ and $2s$. The above equations induce the definition of the quasi-relation

$$\bar{R} =_{\mathbb{E}} [[p_1 \cup p_2, \{s_1, s_2\}], [p_3 \cup p_4, \{s_3, s_4\}], [p_5 \cup \ldots \cup p_{10}, \{s_5, \ldots, s_{10}\}] [p_{11} \cup p_{12}, \{s_{11}, s_{12}\}]],$$

where $qc(p_1 \cup p_2) =_E qc(p_3 \cup p_4) =_E 2$, $qc(p_5 \cup \ldots \cup p_{10}) =_E 6$ and $qc(p_{11} \cup p_{12}) =_E 1$. This quasi-relation $\bar{R}$ is the quasi-set theoretical version for the usual expression $1s^2\ 2s^2\ 2p^6\ 3s^1$.

Now let us turn to quantum statistics. In order to see the differences between the approach of using quasi-sets and the 'standard' form, we shall take another example. Informally speaking, suppose that we have two indistinguishable particles, labelled #1 and #2, distributed among two distinct states, specified by orthogonal wave-functions $\psi_1$ and $\psi_2$.[968] Then, owing

[968] Sant'Anna and Santos 2000 have also considered a number $k > 2$ of particles and a number $n > 2$ of states.

to the indistinguishability of the particles, we know that the corresponding vectors for bosons are

$$|\psi_1^1\rangle \otimes |\psi_2^1\rangle, \; |\psi_2^1\rangle \otimes |\psi_1^1\rangle \text{ and } \frac{1}{\sqrt{2}} \left(|\psi_1^1\rangle \otimes |\psi_2^2\rangle + |\psi_2^1\rangle \otimes |\psi_1^2\rangle\right) \tag{7.6.4}$$

while, in the case of fermions, we have

$$\frac{1}{\sqrt{2}} \left(|\psi_1^1\rangle \otimes |\psi_2^2\rangle - |\psi_2^1\rangle \otimes |\psi_1^2\rangle\right). \tag{7.6.5}$$

So, the labels #1 and #2, originally attached to the particles, were 'veiled' by the appropriate choice of the vectors which express symmetry conditions, as in the usual approach. In order to see how quasi-sets enter in the discussion, let us suppose now that we have defined a quasi-relation $R$ as in equation (7.6.1), that is, $R =_E [\![p, s]\!: p \in \wp \wedge s \in S]$. In other words, we are considering a certain collection of elementary particles (say, particles of several kinds), and let us suppose that $qc(\cup \mathcal{P}) =_E \nu_i$ (this is the total number of particles), subjected to certain states which are taken as elements of a collection $S$ of states. Then, again as above, suppose that we are able to select a family $\{S_i\}_{i \in I}$ of subsets of $S$ such that $\cap_{i \in I} S_i =_E \varnothing$, where $I =_E \{1, 2, 3, \ldots, k_i\}$. Then, for any particular situation $i$, we can define the quasi-relation $R|_i =_E [\![p, s]\!: p \in \wp \wedge s \in S_i]$.

Each relation $R|_i$ describes intuitively a particular distribution of the $\nu_i$ particles in the $k_i$ states of $S_i$. So, consider the fundamental question: *'In how many ways can we distribute $\nu_i$ indistinguishable bosons over $k_i$ quantum states ?'* Since a correspondence between bosons and quantum states is given by the quasi-relations $R|_i$, and taking into account that we are talking about bosons, which are not subject to the restriction imposed by the Pauli Principle mentioned above, the answer is precisely the quantity of quasi-relations $R|_i$ that can be formed.

Let us explain this last assertion by considering a particular case. Suppose that $\nu_i =_E 5$; so we have five indistinguishable bosons to be distributed among, say, three distinct cells (states) $s_1$, $s_2$ and $s_3$. That is, $qc(S_i) =_E 3$. Since collections of bosons with the same quasi-cardinality are indistinguishable in the sense of the Weak Extensionality Axiom, we shall refer to them by their quasi-cardinalities only. Thus, there are 21 different ways of distributing five indistinguishable bosons into states $s_1$, $s_2$ and $s_3$, that is, there are 21 *possible* quasi-relations $R_i^j, j =_E 1, \ldots, 21$ that can be defined for this particular

situation *i*. In the present example, these quasi-relations are shown in the table below, where the number of entries stand for the quasi-cardinality of the quasi-sets $p_1$, $p_2$ and $p_3$ associated with each $s_j$; the numbers 1, …, 21 name the relations $R|_i^1$ to $R|_i^{21}$ :

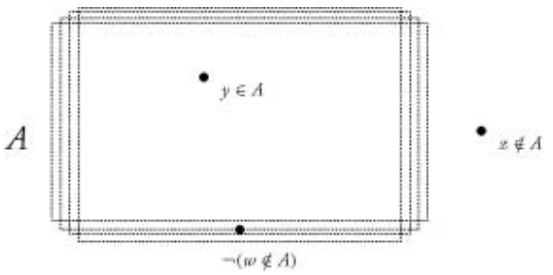


Figure 7.3. The quaset case: The *individual y* belongs to the quaset *A*; the *individual z* certainly does not belong to *A*, and it is false that the individual *w* does not belong to *A* (which does not entail that it belongs to *A*). So, a quaset may be seen as a kind of *fuzzy* collection of *individuals*.

For instance, in the third column, we have the quasi-relation

$$R|_i^3 = {}_E[[p_1,s_1][p_2,s_2][p_3,s_3]], \tag{7.6.7}$$

where $qc(p_1) =_E 4$, $qc(p_2) = 0$ and $qc(p_3) =_E 1$.

This table shows that the number of ways we have to distribute $\nu_i$ indistinguishable bosons in the $k_i$ quantum states *s* is 21, a number which is usually obtained by the Bose-Einstein equation:

$$I^i_{\text{bosons}} = {}_E\frac{(k_i+\nu_i-1)!}{(k_i-1)!\nu_i!}. \tag{7.6.8}$$

This equation expresses here the way of calculating the number of possible relations $R|_i^j$ that can be formed in each situation *i*.[969] In the same way, if we repeat our calculations for fermions; we have, due to the Pauli Principle:

$$I^i_{\text{fermions}} = {}_E\frac{k_i!}{(k_i-\nu_i)!\nu_i!}. \tag{7.6.9}$$

It is important to note that these last equations are usually obtained by considering that different quantum states correspond to distinct energy levels. Our calculations are more general in the sense that we may have different quantum states at the same energy level as in the case of the sodium atom considered above. In interpreting quantum states as energy levels, we could consider, for example, that *S* should be an energy interval given by [0, *kT*], where *k* is the Boltzmann constant and *T* is, say, 300*K*, while each $S_i$ corresponds to an energy range of about $10^{-33}J$ (there are $10^{12}S_i$'s), and each $S_i$ has $10^{19}$ quantum states *s*.

[969] Recall that the numbers involved in the formula are quasi-cardinals of certain qsets.

The total number of microstates corresponding to a given macrostate (energy bin) is given by

$$I_{\text{bosons(fermions)}} =_{E} \prod_i I^i_{\text{bosons(fermions)}}.$$

(7.6.10)

Since classical mathematics can be obtained within the scope of quasi-set theory, all the calculations that follow the derivation of the statistics can be performed here in a straightforward manner. The most probable macrostate will be determined by maximizing $\log I - \alpha N - \beta E$, where $\alpha$ and $\beta$ are Lagrangian parameters that have been introduced to take into account the restriction of fixing the total particle number $N$, total energy $E$, and log is the natural logarithm. Thus, it seems clear that we need to define an injective q-function $f$: $s \rightarrow \mathscr{E}$, where $\mathscr{E}$ is an interval of positive real numbers. Intuitively, $\mathscr{E}$ corresponds to energy.

In the case of fermions we must maximize the following q-function $F$:

$$\begin{aligned} F &=_{\text{df}} \log\left(\prod_i \frac{k_i!}{(k_i-\nu_i)!\nu_i!}\right) - \alpha\sum_i \nu_i - \beta\sum_i \nu_i\varepsilon_i \\ &= \sum_i [k_i(\log k_i - 1) - (k_i-\nu_i)(\log(k_i-\nu_i) - 1) \\ &\quad -\nu_i(\log\nu_i - 1) - \alpha\nu_i - \beta\varepsilon_i\nu_i], \end{aligned}$$

(7.6.11)

where $\varepsilon_i$ stands for the energy associated with each $S_i$. It is clear from these calculations that we have used Stirling's approximation, which states that $\log K! \approx K(\log K - 1)$ for $K \gg 1$. Nevertheless, such an approximation was used just for the bin occupation numbers $\nu_i$ and not for the state occupation numbers.

If

$$\frac{\partial F}{\partial \nu_i} =_{E} 0,$$

(7.6.12)

it follows that

$$\nu_i =_{E} \frac{k_i}{e^{\alpha+\beta\varepsilon_i} + 1}.$$

(7.6.13)

If we assume that the energy differences of the states in the $i$-th bin are negligible, then according to equation (7.6.13), the average occupation of any individual state in that bin is

$$\frac{\nu_i}{k_i} =_{E} \frac{1}{e^{\alpha+\beta\varepsilon_i} + 1}.$$

(7.6.14)

Finally, the average occupation number of the *n*-th single-particle state of energy $\varepsilon_n$ is given by the well-known Fermi-Dirac distribution function:

$$f_{\text{fermions}} =_E \frac{1}{e^{\alpha+\beta\varepsilon_n}+1}. \tag{7.6.15}$$

For bosons, the calculations are very similar, and we have the Bose-Einstein distribution function:

$$f_{\text{bosons}} =_E \frac{1}{e^{\alpha+\beta\varepsilon_n}-1}. \tag{7.6.16}$$

The physical interpretation of the parameters α and β is standard: $\beta =_E 1/kT$, where $k$ is Boltzmann constant and $T$ is the absolute temperature, while α is a normalization constant usually referred to as the *affinity*.

The helium atom is probably the simplest realistic situation where the problem of individuality plays an important role. With identity questions put aside, the wave function of the helium atom would be just the product of two hydrogen atom wave functions with $Z =_E 2$ instead of $Z =_E 1$. Nevertheless, the space part of the wave function for the case where one of the electrons is in the ground state (100) and the other one is in excited state (nlm) is:

$$\phi(\mathbf{x_1}\mathbf{x_2}) =_E \frac{1}{\sqrt{2}}[\psi_{100}(\mathbf{x_1})\psi_{nlm}(\mathbf{x_2}) \pm \psi_{100}(\mathbf{x_2})\psi_{nlm}(\mathbf{x_1})], \tag{7.6.17}$$

where the + (−) sign is for the spin singlet (triplet)[970] and $\mathbf{x}_1$ and $\mathbf{x}_2$ are position vectors of the electrons.

For the ground state, however, the space function needs to be necessarily symmetric. In this case, the problems regarding identity have no physical effect. The most interesting case is certainly the excited state. Equation (7.6.17) reflects our ignorance of which electron is in position $\mathbf{x}_1$ and which one is in position $\mathbf{x}_2$. Nevertheless, in the same equation there are terms like $\psi_{100}\mathbf{x}_1$), which corresponds to a specific physical property of an individual electron.

Our quasi-set theoretical interpretation of equation (7.6.17) is the following (it resembles the case of the sodium atom discussed above). Let $P$ be a pure quasi-set such that $qc(P) =_E 2$. We intuitively interpret the elements of $P$ as electrons of the Helium atom. If $G$ is a unary predicate such that $G(x)$ intuitively says that '$x$ is in the ground state' (the definition of $G$ depends on physical aspects), then, by using the Separation Axiom of $\mathfrak{Q}$, we obtain the

[970] Spin singlet refers to total spin zero and spin triplet refers to total spin different from zero.

sub-quasi-set $p_1 \subseteq P$ defined by

$$p_1 =_{df} [x \in P : G(x)].$$

(7.6.18)

If we call $p_2 =_{df} P - p_1$, then $qc(p_1) =_E qc(p_2) =_E 1$. So, the elements of $p$, despite their indistinguishability, are 'separated' by their 'respective states'. More formally, by calling $g_1$ the ground state and $g_2$ the *other* state, we may define

$$R =_{df} [[p_1, g_1], [p_2, g_2]].$$

(7.6.19)

It is clear that $g_1$ and $g_2$ may be interpreted respectively as 100 and nlm as above. So, we have obtained a way of expressing the fact that given two objects (the elements of $P$) there is one of them in the ground state, although we cannot identify which one, since the qsets $p_1$ and $p_2$ are indistinguishable (in the sense of the Weak Extensionality Axiom). In other words, equation (7.6.19) stands for the situation presented in equation (7.6.17).

We still need to show how the quasi-function $R$ evolves in time, that is, we need to explain the sense according to which equation (7.6.19) plays the role of the wave function given in equation (7.6.17). This may be achieved if we consider the modified quasi-set theory $\mathcal{Q}^t$ described above, in which the quasi-cardinals may vary in time. But this would greatly complicate the mathematical details. Recently, Sant'Anna and Krause have suggested a way of obtaining Schrödinger's equation in $\mathcal{Q}$, but this will be not pursued here, for our aim in this book is not to develop quantum mechanics, but to discuss its mathematical and logical basis.

# 7.7 ON JUSTIFYING QUASI-SET THEORY

Seeking a justification for quasi-set theory may make the reader suspicious, for she could think as Hilbert did that the mathematician (just like the philosopher) is free to investigate all possible theories, and not only those which are close to reality.[971] But, like Gonseth,[972] we could regard logic (here understood in the sense of the 'great logic', that is, involving also set theory) as the physics of any object whatsoever, and hence ask for 'the logic' which underlies certain empirical domains. Taking quantum phenomena as our relevant domain of interest, and following this second direction, we may agree that

[971] Hilbert 1902.

[972] Gonseth 1974.

the basic quantum entities of the same kind may be indistinguishable, and so it is a pertinent question to look for the kind of 'logic' that they obey. In such a logic, of course, we would be able to talk of indistinguishability, and to consider that some entities may have all their relevant properties in common without turning out to be the very same entity, as implied by Leibniz's Law.

Two directions are open to us in this endeavour. The first, and of course the simplest, is that which 'makes physics work'. This is achieved by finding an adequate mathematical description of quantum theory in which the concept of indistinguishability can be dealt with. This route gives us the standard approaches to microphysics. In all the possible ways of obtaining a theory of microphysics which follow this alternative, we need to introduce some kind of principle of symmetry, which (summing up) assumes that permutations of (supposed) indistinguishable entities do not give a distinct situation from that realized before the permutation. An example of this situation is the selection of symmetric and anti-symmetric vectors of the relevant Hilbert space as representing physical phenomena.

But we might be interested in following Post's suggestion that the indistinguishability of quantum entities should be something attributed to them *right at the start*, as a primitive notion, and not obtained *a posteriori* by *ad hoc* devices like those mentioned above (which involve the introduction of some principle of symmetry).[973] This is where quasi-set theory may help. It enables us to consider quantum objects as really and truly indistinguishable entities *right from the start*.

## 7.7.1 Quasi-Sets and Quasets: A New Look

Perhaps the sense of the above comments can be seen once more by considering the differences between quasi-sets and quasets. Figures 7.3 and 7.4 provide an intuitive view of the main distinctions (incorporating what we have considered in Chapter 5 regarding quasets).

Consider the following: let us accept that since no atom contains all electrons of the universe, and since an atom does contain some electrons, then

[973] In Chapter 4, we have made references to the role of the Symmetrization Postulate in this context.

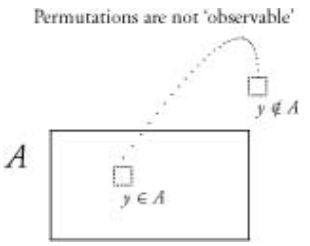


Figure 7.4. The quasi-set case: Either the *non-individual y* belongs to the quasi-set $A$ or it does not, as in the case of an atom, where an electron either belongs or does not belong to it, although we cannot name it unambiguously. Here, $y$ does not act as a name for an individual. Furthermore, permutations of indistinguishable *m*-atoms do not generate 'another' quasi-set, but an indistinguishable one.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $s_1$ | 5 | 4 | 4 | 3 | 3 | 3 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $s_2$ | 0 | 1 | 0 | 0 | 1 | 2 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 5 |
| $s_3$ | 0 | 0 | 1 | 2 | 1 | 0 | 3 | 2 | 1 | 0 | 4 | 3 | 2 | 1 | 0 | 5 | 4 | 3 | 2 | 1 | 0 |

(7.6.6)

there are electrons *inside* the atom and electrons *outside* it.[974] The problem is that we have no criteria for distinguishing them. As Lowe noted,[975] if a neutral atom is ionized, losing one of its electrons and after this 'gains' an electron again, there is no sense in saying that the 'lost' electron is *the same* which was captured to form again a neutral atom. Even talk of 'the same atom' before and after the entire process has to be carefully understood; it would be better to say that the atoms before and after the entire process are *indistinguishable*. Anyway, we know that according to standard physics there are certain numbers

[974] Even if relativistic aspects are considered, where creation and annihilation processes enter the scene, there is a sense in speaking of *the electrons of an atom*, and this possibility suffices for our argument. This will be mentioned again in Chapter 9.

[975] Lowe 1994a.

which express the quantity of electrons in the atom, although we cannot name them (any identification is 'mock', as we have seen). So, the use of individual constants in quasi-set theory to name *m*-atoms is to be avoided.

From the mathematical point of view, let us consider the possibility of obtaining an extension by definition of the theory $\mathfrak{Q}$, by the introduction of a new constant $c$ in its language with the aim of naming an *m*-atom. As is well known, this can be done only if some conditions are obeyed.[976] More specifically, in the case of the introduction of individual constants, some existence conditions of the form $\exists x\,(x = c)$ (which follow from $c = c$ and the substitution axioms of equality),[977] must be proven, and in quasi-set theory, since the predicate of identity does not yield formulas when *m*-atoms are considered, these conditions cannot be provided. To summarize: we cannot obtain an extension by the usual definition of $\mathfrak{Q}$ if we try to add to its language an individual constant for naming *m*-atoms. In order to introduce new constants in the language of $\mathfrak{Q}$, of course identity cannot be used, but we could modify the standard procedure by using for instance an adequate congruence; our relation of indistinguishability '$\equiv$' may do the job, but then the new constants would not distinguish among the elements of a whole collection of objects (the equivalence classes of congruent objects).

But, a final question could still be posed: can we name *m*-atoms *from the outside*? This is an interesting question, which can be answered as follows. The term *outside* here stands for some 'naming-function' which would attribute to an *m*-atom a certain *name*, say the number 1, but such that it cannot be proved to exist *in* $\mathfrak{Q}$.[978] This identification, similar to that which physicists intuitively use when they 'name' elementary particles by $n_1$, $n_2$, …, is only apparent (say, something made in the metalanguage), and any identification is lost when symmetry conditions are introduced in order to make these labels otiose. So, as we said before, owing to the symmetry conditions, only in *mente Dei* could the *m*-atoms be said to have genidentity, or individuality.

[976] Shoenfield 1967, Section 4.6.

[977] Ibid., p. 60.

[978] Again, we are reminded of Skolem's 'paradox'; see Fraenkel et al. 1973, pp. 302 ff.

# 8
# Non-Reflexive Quantum Logics

> Si nous envisageons des domaines qui diffèrent du monde familier à notre échelle, les lois existentielles changent dans la mesure où les propriétés des nouveaux objets abordès changent elles-mêmes, et nous avons affaire à d'autres logiques.
>
> P. Destouches-Fèvrier 1951, p. 5

In this Chapter we shall present a class of logical systems for which the Principle of Identity is not generally valid. Since one of the versions of this principle which will be considered is (in a first order language) $\forall x\,(x = x)$ (this expression is also termed 'the reflexive law of identity'), we call such logics *non-reflexive*, as noted in the previous chapter.[979] Furthermore, since the basic motivation for these systems is the treatment given to quantal objects, it can also be said to be a kind of *quantum logic*. In addition, it is a quantum logic distinct from the standard systems presented in the literature,[980] which deal with certain kinds of lattices, as we also noted in the previous chapter.

The non-reflexive systems to be presented here are based on Schrödinger's intuitions discussed earlier in this book and are called *Schrödinger Logics*. We begin by presenting the first-order system *S* suggested by Newton da Costa and then we turn to the corresponding higher-order systems. Considerations regarding the semantics of such systems underpin the understanding of the role played by classical set theories (like ZF) in such a semantics. The limitations of such a 'classical' semantics motivate the development of a semantics for Schrödinger logics (involving modalities)

[979] More generally, we shall denote by 'non-reflexive' any logical system which *deviates* from the 'traditional' theory of identity discussed in Chapter 6, in particular by violating the Principle of Identity in one of its forms. For a more general discussion on these logics, see Krause and Béziau 1995.

[980] A general and detailed text on quantum logics is Dalla Chiara and Giuntini 2001; see also Dalla Chiara, Giuntini and Greechie 2004.

founded in quasi-set theory. In the last section, we discuss a possible relationship between these formal systems and the general idea of sortal predication as applied to quantum physics. Since in higher-order logics identity can be defined, our systems provide a formal framework for expressing the status of the Principle of the Identity of the Indiscernibles as well.

# 8.1 MOTIVATION

As we have already noted, following Schrödinger's suggestion regarding identity in the quantum domain, da Costa outlined a two-sorted first-order logical system $S$ in which the principle of identity $\forall x(x = x)$ is not valid in general.[981] A two-sorted first-order language was proposed, with a primitive symbol = of identity, encompassing variables of first species, say $x$, $y$, … (and possibly constants of the first species) and of second species, say $X$, $Y$, … (constants of second species might also be considered). In the definition of well-formed formulas, expressions like t=u were taken to be well formed if and only if both $t$ and $u$ were terms of second species only (these details will be covered below); for the terms of the first species, that expression is not considered to be a well-formed formula. The apparently strange idea of 'not having' a way of expressing some fact can be grasped if we consider a certain assertion about the angle between two vectors (for instance, that they are orthogonal). We cannot formulate such a claim in the language of vector spaces without an inner product. In order to say that two vectors are orthogonal, we need to expand the vector space structure to include adequate tools for expressing metric concepts. The same holds in the present case: the proposed language imposes a restriction on the concept of formula, and this is what (in our approach) makes sense of Schrödinger's intuitions.

In other words, using the proposed language, one cannot talk about either the identity or the diversity of those entities denoted by the terms of the first species. So, when we say that $\forall x(x = x)$ *is not valid in general*, this does not mean that we will present a kind of object which is distinct from itself, but that the employed language has no formal tools for expressing talk of either the identity or the diversity of all objects of the domain.

[981] da Costa 1980, pp. 119 ff; 1997, pp. 123 ff.

The axioms of the logic *S* were taken by da Costa to be essentially those of first-order (many-sorted) logic, but with regard to the axioms of identity, he assumed that they hold only for entities of the second species, owing to the imposed restriction on the concept of formula. It was also suggested that we could develop a semantics for such a system. However, when describing such a semantics, if standard set theories (like Zermelo-Fraenkel, ZF) are used, although the relevant results can be achieved, such as a completeness theorem (see below), from the philosophical point of view something appears to be wrong with respect to the basic intuitions: according to the spirit of Schrödinger logics, the terms of first species should be interpreted as elements for which the concept of identity does not make sense. But, if the domain of the interpretation is taken to be a *set* (in ZF, say), we can't express this idea. In considering this fact, da Costa suggested that the very notion of 'set' should be generalized by the development of a theory of *quasi-sets*, as outlined in the previous chapter and which should encompass standard sets as particular cases.

This first-order system was further extended to a higher-order logic $S_\omega$ (simple theory of types, where ω stands from now on for the set of whole numbers).[982] The idea behind such an extension to higher-order logics was to explore the system as a 'mathematical system' properly speaking, in the sense that higher-order logics enable us to develop much of standard mathematics, but also (and mainly) to explore the philosophical issues associated with finding an expressive language in which we may regard the Principle of the Identity of Indiscernibles as inapplicable, in the sense we have already discussed.

The reason it is not so straightforward to construct such a language is that in higher-order logics the concept of identity is typically defined by means of indistinguishability, that is, expressions like $x = y$ can be taken as abbreviations for $\forall F(F(x) \leftrightarrow F(y))$ (noting the restrictions on the types of the involved terms, as we shall make clear below). The challenge was to find a way of having the latter (indistinguishability) but without giving meaning to the former (identity). The solution we have found, which of course is just one among various possible alternatives, is to consider Schrödinger Logics. Hence, owing to the restriction on the concept of formula, we can say that certain objects may have all their attributes in common (indistinguishability), but this does not entail that they should be the same entity (numerical identity).

[982] Krause 1990; see da Costa and Krause 1994.

The proposed higher-order system $S_\omega$ was further extended in order to encompass modalities for expressing intensional concepts as well, and a semantics based on quasi-set theory was also outlined.[983] This intensional system will be presented below.

## 8.2 FIRST-ORDER SYSTEMS

Da Costa's system $S$ is a two-sorted first-order logic which has the following categories of primitive symbols: (a) connectives: $\neg$ (not) and $\vee$ (or) (the other ones are defined as usual); (b) the universal quantifier: $\forall$ (for all) (the existential quantifier is defined as usual); (c) parentheses and comma; (d) a denumerably infinite set of individual variables of first species $x_1$, $x_2$, … and a set of individual constants of that species: $a_1$, $a_2$, …; (e) a denumerably infinite set of individual variables of second species: $X_1$, $X_2$, … and a set of individual constants of second species: $A_1$, $A_2$, …; (f) the symbol of equality, =; (g) for each natural number $n > 0$, a non-empty collection of $n$-ary predicate symbols. We use $x$, $y$, etc. as syntactical variables for terms in general, and the context will avoid any confusion.

By a *term* we understand any variable or constant; hence, we have terms of first species (which we call here *m*-terms, in agreement with the terminology of quasi-set theory) and terms of the second species (called *M*-terms). Intuitively, the *m*-terms may be thought of as denoting quantum objects, while the *M*-terms stand for the macroscopic ones. With respect to this last category, we suppose that classical logic is valid in all its aspects. The *formulas* of $S$ are defined in the standard way,[984] but note that $x = y$ will be considered as a formula if and only if $x$ and $y$ are terms of second species.

Since the expression $x = y$ is not a formula if either $x$ or $y$ are *m*-terms (terms of first species), the language of $S$ does not permit us to talk about either the identity or the diversity of the *m*-objects (entities denoted by the *m*-terms). The other syntactic concepts are the usual ones. The propositional postulates (schema of axioms and inference rules) for $S$ are also standard, based on the primitive connectives we have selected.[985] When it comes to quantification and

[983] Krause 1995; da Costa and Krause 1997.

[984] Wang 1952.

[985] Hilbert and Ackermann 1950/1999, Chapter 4.

identity, the postulates are:

(S1) $\forall x\alpha(x) \rightarrow \alpha(t)$, where $x$ is a variable and $t$ a term of the same species of $x$ which is free for $x$ in $\alpha(x)$.
(S2) $\beta \rightarrow \alpha(x)/\beta \rightarrow \forall x\alpha(x)$ where $x$ does not occur free in $\beta$.
(S3) $\forall x(x = x)$ where $x$ is an $M$-variable.
(S4) $t = u \rightarrow (\alpha(t) \leftrightarrow \alpha(u))$ with the usual restrictions, provide that $t$ and $u$ are both $M$-terms.

The notions of theorem, syntactic consequence of a set of formulas etc. are also standard. Following da Costa's suggestions, a 'classical' semantics for this system (that is, a semantics founded in a standard theory of sets like ZF) can be outlined as follows: we interpret the language of $S$ in a structure whose domain is a set $D = D_1 \cup D_2$, with $D_1 \cap D_2 = \varnothing$ as usual, with the $n$-ary predicates we associate subsets of $D^n$. The $m$-constants are associated with elements of $D_1$, while the $M$-constants denote elements of $D_2$. With the equality symbol, we associate the diagonal of $D_2$, namely, the set $\Delta_{D_2} = \{\langle x,x\rangle : x \in D_2\}$. The semantical results can be obtained without difficulty.

Of course, as da Costa has pointed out, strictly speaking $D_1$ should not be considered as a *set* in the usual sense, since the predicate of identity has sense only in connection with the elements of $D_2$. In other words, the metamathematics employed to build the structure where the language of $S$ is interpreted does not agree with the underlying motivation of the logic. Da Costa thus proposed that a *theory of quasi-sets* should be developed, in order to provide a semantics for $S$, as presented in the previous chapter.

Since our concern is with aspects of quantum theory, we shall begin by extending the system $S$ to a higher-order logic $S_\omega$ in which, for instance, a version of the Principle of the Identity of Indiscernibles can be formally expressed. Thus, in the next section we will present a Schrödinger logic of order $\omega$ (simple theory of types) and we will sketch its Henkin semantics, while still working within ZF. Then we shall turn again to a discussion of the nature of such a semantics and, in order to built a semantics based on the theory $\mathfrak{Q}$ as presented in the last chapter, we will further enlarge the theory $S_\omega$ by adding modal operators to its language, so obtaining the theory $S_\omega\mathscr{J}$ in which certain intensional concepts can be dealt with. This will enable us to consider that in quantum theory there

may be 'predicates' which do not have a well-defined extension. Furthermore, these latter systems will be used to discuss certain ideas involving sortal predication.

# 8.3 HIGHER-ORDER SCHRÖDINGER LOGICS

The system we present here, denoted by $S_\omega$, is a natural extension of the first-order logic *S* to the simple theory of types; therefore, we will not describe all the details concerning its language, which contains the same connectives, the equality symbol and auxiliary symbols as *S*. In order to present its variables and constants, we begin by defining the set of *types* as the smallest set Π such that: (a) $i \in \Pi$ (*i* is the type of the *individuals*) and (b) if $\tau_1, \ldots, \tau_n$ belong to Π, then $\langle \tau_1, \ldots, \tau_n \rangle \in \Pi$.

Then, for each type $\tau \in \Pi$ ($\tau \neq i$), there exists a denumerably infinite set of variables $X_1^\tau, X_2^\tau, \ldots$ of type τ and a set of constants $A_1^\tau, A_2^\tau, \ldots$ of that type. When $\tau = i$, there are two sorts of terms: the *M*-terms and the *m*-terms. The former are the variables $X_1^i, X_2^i, \ldots$ and the constants $A_1^i, A_2^i, \ldots$ called *M*-variables and *M*-constants respectively, whereas the latter are the *m*-variables $x_1^i, x_2^i, \ldots$ and the *m*-constants $a_1^i, a_2^i$. In other words, we have a two-sorted language at the level of objects. We use $U^\tau, V^\tau, \ldots$ and $u, v, \ldots$ as syntactic variables for terms of type τ (including the *M*-terms of type *i*) and for *m*-terms respectively; $U, V, \ldots$ are used as syntactic variables for any terms in general.

The definition *terms* of type τ and of *atomic formulas* are standard, but with respect to the predicate of identity, we require that only expressions of the form $U^\tau = V^\tau$ are atomic formulas; hence, expressions such as $U^\tau = u$, or $u = v$ etc. are meaningless. Consequently, again, we cannot talk about either the identity or the diversity of the objects denoted by the *m*-terms. The postulates of this logic are similar to those of the standard higher-order systems, including the axioms of Extensionality, Separation and Infinity.[986] The case of the Axiom of Choice will be discussed below. With respect to equality, we introduce the following *Axiom of Equality*:

$$U^\tau = V^\tau \leftrightarrow \forall F^{\langle\tau\rangle}(F^{\langle\tau\rangle}(U^\tau) \leftrightarrow F^{\langle\tau\rangle}(V^\tau)), \tag{8.3.1}$$

[986] Hilbert and Ackermann *loc. cit.*

where $U^{\tau}$ and $V^{\tau}$ are terms of type τ, but not *m*-terms, and $F$ is a variable of type $\langle\tau\rangle$.

**Definition 30 (Absolute Indistinguishability).** *If $U$ and $V$ are any terms of type τ (including m-terms of type i) and $F$ is a variable of type $\langle\tau\rangle$, then*

$$U \equiv V =_{\mathrm{df}} \forall F(F(U) \leftrightarrow F(V))$$

If $U \equiv V$, we say that $U$ and $V$ are *absolutely indistinguishable*. Note that the definition holds also for *m*-terms, since it does not exclude the possibility that $\tau = i$. Hence, by the definition of the atomic formulas just mentioned, there is a sense in which, according to the canons of $S_{\omega}$, the entities denoted by the *m*-terms may be 'absolutely indistinguishable', without being identical. As a consequence, Leibniz's Law does not hold in general. In addition, let us remark that from the axiom and definition above, it follows that if $U$ and $V$ are terms of type $\langle\tau\rangle$, but not *m*-terms, then $U \equiv V$ is equivalent to $U = V$, that is to say, *identity* and *indistinguishability* are equivalent for those entities which are not *m*-terms. In other words, the traditional theory of identity remains valid in the 'macroscopic world'.

**Definition 31 (Relative Indistinguishability).** *If $U$ and $V$ are terms of type τ, $F$ is a variable of type $\langle\tau\rangle$ and $P_{\tau}$ is a constant of type $\langle\langle\tau\rangle\rangle$, then*

$$U \equiv_{P_{\tau}} V =_{\mathrm{df}} \forall F(P_{\tau}(F) \rightarrow (F(U) \leftrightarrow F(V))$$

If $U \equiv_{P_{\tau}} V$, we say that $U$ and $V$ are indistinguishable *with respect to*, or *relative to* the attributes 'characterized' by $P_{\tau}$. $U$ and $V$ being relatively indistinguishable means only that they agree with respect to some class of attributes. It is interesting to note that the concept of relative indistinguishability can be formulated also within classical higher-order logic in exactly the same way. By using definition (31), we can formulate the concept of indistinguishable particles as used in quantum mechanics.[987] This notion can be expressed in our formalism by using the above definition and considering a predicate $I$ of type $\langle\langle\tau\rangle\rangle$ such that $I(F)$ says intuitively that $F$ is an intrinsic property (whatever this means, since this is not important from the formal point of view). Then, to say that $U$ and $V$ are 'indistinguishable particles' in quantum mechanics means $U \equiv_I V$, but of course not that $U = V$. We shall return to these points in the last section.

[987] Jauch 1966, p. 275.

The concepts of both absolute and relative indistinguishability can be related by means of the following result, which can be easily proved as a theorem of $S_\omega$: $\forall F(P_\tau(F) \rightarrow (U \equiv_{P_\tau} V \leftrightarrow U \equiv V))$.

The usual axiom of choice can also be easily adapted to our case by using the relation of absolute indistinguishability instead of equality.[988] So, if $F$ and $F'$ are variables of type $\tau_1$, $G$ and $L$ are variables of type $\tau_2$, $A$ and $H$ are variables of type $\langle \tau_1, \tau_2 \rangle$, the following expression is an axiom of $S_\omega$ (the Axiom of Choice):

$$\forall A \exists H(\forall F(\exists G A(F,G) \rightarrow \exists G(H(F,G) \wedge A(F,G))) \rightarrow \forall F \forall F' \forall G \forall L(H(F,G) \wedge H(F',L) \wedge F \equiv F' \rightarrow G \equiv L)).$$

(8.3.2)

In other words, $H$ 'selects' indistinguishable objects from the collection of the images (by $A$) of indistinguishable objects (the $F$'s). It is obvious that if we are not considering *m*-terms, the symbol ' = ' can replace '≡' (by the above results) and the above expression turns out to be equivalent to the axiom used in the standard simple theory of types.

It is possible to define a translation from the language of the simple theory of types to the 'macroscopic part' of the language of our system by supposing that all variables and constants that occur in the formulas are *M*-variables or *M*-constants. Then, we can prove the following theorem, which states that all the results which can be obtained in the simple theory of types can also be obtained in $S_\omega$.

**Theorem 32.** *Let* $\alpha$ *be a formula of the simple theory of types and* $\alpha_\omega$ *its translation in the language of* $S_\omega$. *Then, if* $\alpha$ *is a theorem of the simple theory of types,* $\alpha_\omega$ *is a theorem of* $S_\omega$. *So, all the mathematics that can be developed in the simple theory of types can also be developed in* $S_\omega$.

## 8.3.1 Identity and Absolute Indistinguishability

The reader may (justifiably) feel unsatisfied with the above definition of absolute indistinguishability. Perhaps this is because it does not essentially differ from the standard definition of the predicate ' = ' of identity in type theory (the Leibniz Law). The only difference is that we have used the symbol ≡ instead of =. So, one might say, we are only changing the name of the defined predicate and just saying that the agreement with respect to all properties has a name

[988] Hilbert and Ackermann op. cit., p. 156.

other than 'identity', that is, 'absolute indistinguishability', but formally they are the same concept.

From the syntactic point of view, this is in fact the case, for instead of our Axiom of Equality (8.3.1), we could give another definition stating that (here we shall leave the types implicit)

$$U = V =_{\mathrm{df}} \forall F(F(U) \leftrightarrow F(V)), \tag{8.3.3}$$

and from the syntactic point of view nothing can distinguish between $U \equiv V$ and $U = V$ (really, $U \equiv V$ if and only if $U = V$). The distinction between these concepts can be seen from the semantic point of view only, for (according to the standard patterns) $U = V$ says that $U$ and $V$ have the same referent, while $U \equiv V$ says (according to our interpretation) that the objects denoted by $U$ and by $V$ differ *solo numero*. In order to see the difference, that is, in order to make sense of this second concept without collapsing it into the former, we need to turn to the semantics of Schrödinger logics.[989]

# 8.4 A 'CLASSICAL' SEMANTICS FOR $S_\omega$

In this section, we outline a proof of a 'weak' completeness theorem for $S_\omega$ in the sense of Henkin.[990] In what follows, the language of $S_\omega$ will be denoted by $\mathscr{L}$.

If $\mathbf{D}$ is an infinite set such that $\mathbf{D} = \mathbf{m} \cup \mathbf{M}$ and $\mathbf{m} \cap \mathbf{M} = \varnothing$,[991] then a *frame* for $\mathscr{L}$ based on $\mathbf{D}$ is a function $\mathscr{M}$ whose domain in the set $\Pi$ of types such that $\mathscr{M}(i) = \mathbf{D}$ and, for each type $\tau = \langle \tau_1, \ldots, \tau_n \rangle \in \Pi, \mathscr{M}(\tau) \subseteq \mathcal{P}(\mathscr{M}_{\tau_1} \times \ldots \times \mathscr{M}_{\tau_n})$. If the inclusion in this last expression is replaced by the equality symbol, than the frame is *standard*.

[989] Of course other ways of dealing with this problem can be analysed, such as, for instance, by examining the meaning of the universal quantifier in the above definitions. This of course makes sense, since perhaps the use of quantifiers in physics does not strictly agree with the 'objectual' interpretation, according to which $\forall x A(x)$ is true if and only if the condition $A$ is satisfied by each element of the domain. In other words, 'classical' quantifiers are compromised by sets due to the standard semantics. Perhaps within the domain of physics we could define a kind of 'operational quantifier', relative to the properties the physicist really deals with within a certain framework, so that $\forall x A(x)$ would be true if and only if each substitutional instance of $A$ by 'operational properties' is true. But we shall not pursue this idea here, since this interpretation of the quantifiers can be mapped into our definition of relative indistinguishability (in this case, we should always read 'relative' as '*relative* to operational properties').

[990] We follow Church 1956, § 54, but part of our terminology follows Rogers 1971, Chapter IV.

[991] $\mathbf{D}$ is taken to be infinite since we intend to consider models also for the axiom of infinity.

If we write $\mathcal{M}_\tau$ instead of $\mathcal{M}(\tau)$, then the frame can be viewed as a family $(\mathcal{M}_\alpha)_{\alpha\in\Pi}$ of sets satisfying the above conditions. In what follows, we will refer to both this family and the set $\mathcal{F}_{\mathbf{D}} = \{X\colon \exists\, \tau \in \Pi \wedge X = \mathcal{M}(\tau)\}$ indifferently as *the frame* for $\mathcal{L}$ based on **D**.

A *denotation* for $\mathcal{L}$ based on **D** is a function $\varphi$ whose domain is the set of constants of $\mathcal{L}$, defined as follows:

(i) $\phi(a_j^i) \in \mathbf{m}, j = 1,2,\ldots$, $j$ = 1, 2, …
(ii) $\phi(A_j^i) \in \mathbf{M}, j = 1,2,\ldots$, $j$ = 1, 2, …
(iii) $\phi(A_j^\tau) \in \mathcal{M}_\tau$ for every $\tau \neq i$, $j$ = 1, 2, ….

In particular, $\varphi(=) \in \mathcal{M}_{\langle\tau,\tau\rangle}$ (recall that the symbol ' = ' is written ambiguously to denote a predicate of type $\langle\tau, \tau\rangle$). By means of this definition, we introduce the notion of an *interpretation* for $\mathcal{L}$ based on **D** as an ordered pair $\mathcal{A} = \langle(\mathcal{M}_\alpha)_{\alpha\in\Pi}, \varphi\rangle$, where $(\mathcal{M}_\alpha)_{\alpha\in\Pi}$ and $\varphi$ are as above. The interpretation is *principal* as the frame is standard and $\varphi(=) = \Delta\mathcal{M}_\tau$, the diagonal of $\mathcal{M}_\tau$.

A *valuation* for $\mathcal{L}$ (based on $\mathcal{A}$) is a function $\psi$ whose domain is the set of terms of $\mathcal{L}$ such that $\psi$ is an extension to the set of terms of $\mathcal{L}$ of the denotation $\varphi$. In other words, $\psi$ is defined as follows:

(i) $\psi(t) = \varphi(t)$ if $t$ is a constant
(ii) $\psi(x_j^i) \in \mathbf{m}$ for the $m$-variables ($j$ = 1, 2, …)
(iii) $\psi(X_j^i) \in \mathbf{M}$, for the $M$-variables ($j$ = 1, 2, …)
(iv) $\psi(X_j^\tau) \in \mathcal{M}_\tau$ for $\tau \neq$ i ($j$ = 1, 2, …).

The definition of *satisfiability*, that is, the concept of $\mathcal{A}, \psi \vDash A$, is defined by recursion on the length of the formula $A$ as in the standard case. So, if $\mathcal{A}$ is an interpretation of $\mathcal{L}$ based on **D**, then

(i) $\mathcal{A},\psi \vDash F(X_1^\tau \ldots X_n^\tau)$ iff $\langle\psi(X_1^\tau),\ldots,\psi(X_n^\tau)\rangle \in \psi(F)$, where $F$ is a term of type $\langle\tau_1, \ldots, \tau_n\rangle$ and the $X_n^\tau$ are terms of type $\tau_j (j = 1, \ldots, n)$.
(ii) $\mathcal{A}, \psi \vDash U = V$ iff $\langle\psi(U), \psi(V)\rangle \in \psi(=)$, where $U$ and $V$ are both terms of same type $\tau$.

(iii) The satisfaction clauses for ¬, ∨ and ∀ are introduced as usual.
If $A$ is an instance of the axioms of Extensionality, Separation, Choice or Infinity and $\mathscr{A}$ is an interpretation for $\mathscr{L}$ based on **D**, then $\mathscr{A}$ is an *appropriate interpretation* for $\mathscr{L}$ iff $\mathscr{A}, \psi \vDash A$. In what follows, we will consider only appropriate interpretations.

An (appropriate) interpretation is *normal* iff $\mathscr{A}, \psi \vDash A$ where $A$ is either an axiom of $S_\omega$ or is derived by means of the inference rules from formulas $B_1, \ldots, B_n$ of $\mathscr{L}$, and $\mathscr{A}, \psi \vDash B_j, j = 1, \ldots, n$. A normal interpretation for $\mathscr{L}$ which is not a principal interpretation is said to be a *secondary* interpretation.

A formula $A$ is *true* with respect to an interpretation $\mathscr{A}$ iff $\mathscr{A}, \psi \vDash A$ for every valuation $\psi$ based on $\mathscr{A}$. $A$ is *valid* iff it is true with respect to all principal interpretations; $A$ is *satisfiable* iff there exists a valuation $\psi$ and a principal interpretation $\mathscr{A}$ such that $\mathscr{A}, \psi \vDash A$. $A$ is *secondarily valid* iff it is true under all normal interpretations. Finally, $A$ is *secondarily satisfiable* iff there is a valuation $\psi$ with respect to a normal interpretation $\mathscr{A}$ such that $\mathscr{A}, \psi \vDash A$.

The following results can be proved without difficulty: (1) $A$ is valid iff $\neg A$ is not satisfiable; (2) $A$ is secondarily valid iff $\neg A$ is not secondarily satisfiable; (3) $A$ is satisfiable iff $\neg A$ is not valid; (4) $A$ is secondarily satisfiable iff $\neg A$ is not secondarily valid and (5) $A$ is valid (respectively secondarily valid) with respect to a normal interpretation iff its universal closure is valid (respectively secondarily valid) with respect to this interpretation.

If $\Gamma$ is a set of formulas of $\mathscr{L}$, then a *model* of $\Gamma$ is a normal interpretation $\mathscr{A}$ such that $\mathscr{A}, \psi \vDash A$ for every formula $A \in \Gamma$. If $\mathscr{A}$ is a principal interpretation, we will talk of *principal models*, or of *secondary models* if $\mathscr{A}$ is a secondary interpretation. The following terminology will be used below: $\Gamma \vDash A$ means that $A$ holds in every model of $\Gamma$, and $\vDash A$ means that $A$ is secondarily valid.

The following results can then be proven:[992]

**Theorem 33(Soundness).** *All theorems of $S_\omega$ are secondarily valid. Hence, they are valid.*

That is, $\vdash A$ implies $\vDash A$; it is not difficult to generalize this result: $\Gamma \vdash A$ entails $\Gamma \vDash A$.

[992] As in Church *loc. cit.*

**Lemma 34 (Lindenbaum).** *Every consistent set $\Gamma$ of closed formulas of $\mathscr{L}$ can be extended to a maximal consistent class $\overline{\Gamma}$ of closed formulas of $\mathscr{L}$ (the concepts introduced here are the usual ones).*

We can now state a basic lemma:

**Lemma 35.** *If $A$ is a closed formula of $\mathscr{L}$ which is not a theorem, then there exists a normal interpretation whose domains $\mathscr{M}_\alpha$ are denumerably infinite, with respect to which $\neg A$ is valid.*

**Proof.** Let $\mathscr{L}'$ be the language obtained by adding to $\mathscr{L}$ the following list of symbols: (a) two disjointed denumerable infinite sets of new constants of type $i$, $w_1$, $w_2$, …, and $W_1$, $W_2$, …. We shall say that the first set is a new set of $m$-constants and that the second one is a new set of $M$-constants. (b) for each $\tau \in \Pi(\tau \neq i)$, a denumerably infinite set of new constants of type $\tau, W_1^\tau, W_2^\tau, \ldots$.

We still suppose that there exists a fixed enumeration of the closed formulas of $\mathscr{L}'$. Let $H$ be a closed formula of $\mathscr{L}$ which is not a theorem of $S_\omega$. Then we define recursively the classes $\Gamma_j$, $j = 1, 2, \ldots$ as follows:

(a) $\Gamma_0 =_{df} \{\neg H\}$

(b) If the $(n + 1)$th closed formula of $\mathscr{L}'$ has the form $\forall X^\tau A$, and if the first new constant of type $\tau$ which does not occur either in $A$ or in any member of $\Gamma_n$ is $W_m^\tau$, then $\Gamma_{n+1}$ is $\Gamma_n$ plus all expressions of the form

$$Subst(W_m^\tau, X^\tau, A) \rightarrow \forall X^\tau A,$$

where $Subst(W_m^\tau, X^\tau, A)$ is the result of substituting $W_m^\tau$ for all free occurrences of $X^\tau$ in $A$. Otherwise, $\Gamma_{n+1}$ is $\Gamma_n$. We note that if $X^\tau$ is a $m$-variable of type $i$, then $W_m^\tau$ must be taken from the list $w_1$, … of new $m$-constants. As in Church,[993] we may prove that the $\Gamma_j$ ($j = 1, 2, \ldots$) are consistent. Then, we define $\Gamma =_{df} \cup_j \Gamma_j$ and let $\overline{\Gamma}$ be a maximal consistent class in the sense of Lindenbaum's Lemma above.

Now let $\mathbf{D} = \mathbf{m} \cup \mathbf{M}$ be a set as in the above definition of a frame for $\mathscr{L}$. Then we define a frame for $\mathscr{L}'$ based on $\mathbf{D}' = \mathbf{m}' \cup \mathbf{M}'$, where

[993] Church, op. cit., pp. 311–12.

$m' =_{df} m \cup \{t_1, t_2, \ldots\}$ and $M' =_{df} M \cup \{t'_1, t'_2, \ldots\}$, with $\{t_1, t_2, \ldots\} \cap \{t'_1, t'_2, \ldots\} = \varnothing$, where both sets $\{t_1, t_2, \ldots\}$ and $\{t'_1, t'_2, \ldots\}$ are denumerably infinite. The frame is then defined as follows: (a) $\mathscr{M}_i =_{df} D'$, and (b) $\mathscr{M}_\tau =_{df} \{\varphi(F)$: $F$ is a constant of $\mathscr{L}'\}$, and $\varphi$ is an application whose domain is the set of constants of $\mathscr{L}'$ defined in the following way, where $j = 1, 2, \ldots$: (1) $\phi(a_i^j) \in m$ (2) $\phi(w_j = t_j)$ (3) $\phi(A_i^j) \in M$ (d) $\phi(W_j^i) = t_j'$ (e) for each constant $F$ of type $\tau = \langle \tau_1, \ldots, \tau_n \rangle \neq i$, $\varphi(F) =_{df} \{\langle \varphi(T_1), \ldots, \varphi(T_n) \rangle \in \mathscr{M}_{T_1} \times \ldots \times \mathscr{M}_{T_n} : F(T_1 \ldots T_n) \in \overline{\Gamma}\}$, where the $T_j$ are constants of types $\tau_j$. Then, we can prove (1) that every $\mathscr{M}_\tau$ is denumerable and (2) that $\mathscr{A} = \langle (\mathscr{M})_{\tau \in \Pi}, \varphi \rangle$ is a model for every formula $A$ of $\mathscr{L}$. Really, if $X_1, X_2, \ldots, X_n$ are variables occurring in $A$ of types $\tau_1, \tau_2, \ldots$ respectively, then $\mathscr{A}, \varphi \vDash A$ if and only if the formulas obtained by replacing $T_1, T_2, \ldots$ for all occurrences of $X_1, X_2, \ldots$ in $A$ belong to $\overline{\Gamma}$. So, let $\neg H$ be such a formula. Then it belongs to $\overline{\Gamma}$ since this set is complete and by hypothesis $H$ is not a theorem. Hence, $\neg H$ is valid with respect to $\mathscr{A}$ above, and this proves the lemma.

By using this lemma, we can prove our main result:

**Theorem 36 (Henkin Completeness).** *Every formula of $S_\omega$ which is secondarily valid is a theorem.*

**Proof.** Let $A$ be a secondarily valid formula of $\mathscr{L}$ and let $H$ be its universal closure. Then, by the above Lemma, $H$ is secondarily valid. But, in this case, $H$ is true with respect to all sound interpretations, hence $\neg H$ is not secondarily satisfiable, which entails that there exists an interpretation relative to which $\neg H$ is valid. So, by the Fundamental Lemma, $H$ is a theorem, hence $A$ is a theorem.

In other words, $\vDash A$ implies $\mathscr{J} A$. In general, if $\Gamma$ is a set of closed formulas of $\mathscr{L}$ which is not inconsistent, then $\Gamma \vDash A$ implies $\Gamma \vdash A$, that is, if $A$ holds in every model of $\Gamma$, then $A$ is derivable from the formulas of $\Gamma$.

In the above, in taking $\mathbf{D} = \mathbf{m} \cup \mathbf{M}$ as a *set*, we keep the semantics, subject to the same problems already alluded to with respect to the first-order systems. Of course, from the point of view of $S_\omega$, **m** should be not considered as a set, since in principle the relation of equality cannot be applied to its elements. So the problem remains of basing a semantics for Schrödinger logics on quasi-set theory. We shall solve this problem below, but in connection with a modified

version of the system presented above. Before we do that, let us reinforce certain points regarding this semantics.

### 8.4.1 Identity and Indistinguishability Revisited

Given that we can *define* identity via Leibniz's Law, as we have suggested in section 8.3.1, there is apparently no distinction between such a defined concept and absolute indistinguishability as captured by definition 30. As we have suggested, such a distinction can be established only from a semantic point of view, since it cannot be given syntactically (that is, within the language of $S_{\omega}$ ). But the above semantics does not help us in expressing the idea that we have in the domain objects which differ *solo numero* (absolutely indistinguishable) but which are not the very same object. The reason is the same as that given above concerning first-order Schrödinger logics, namely, that since the domain of the interpretation is a standard *set*, its elements are always (yet potentially) distinguishable from one another, and this runs against Schrödinger's intuitions, as we have seen. Thus, we need to find a way of interpreting the *m*-constants of type *i* so that their referents, under certain conditions, can be assumed to be absolutely indistinguishable. This may be achieved by a quasi-set semantics, as we shall see below. However, it would be useful, of course, to not only develop such a semantics, but enrich the systems in order to accommodate the formal discussion of other aspects of quantum entities. This is why we shall consider a modified version of the higher-order system presented above. Let us now turn to the details.

## 8.5 THE INTENSIONAL SYSTEM $S_{\omega\mathcal{I}}$

In quasi-set theory a separation schema is assumed, so we may talk of the sub-qsets of a given qset, and in considering the intended interpretation of 'pure' qsets as collections of quanta, we may talk of collections of indistinguishable objects obeying certain conditions. Consider the following example: Suppose we are considering the four neutrons of the nucleus of a $^7$Li atom. We may suppose the existence of sub-collections of two of them; call *P* the property which expresses that fact. Intuitively speaking (by applying elementary mathematics), there are six sub-collections with cardinal 2 each. But this counting is made under the hypothesis that the neutrons are distinct from each other,

which of course is not the case. If we want to specify the extension of the predicate *P* above, according to standard semantics, what sub-collection should we choose? There is no criterion of choice. We express this by saying that the extension of *P* is not well defined, an idea already expressed by Dalla Chiara and Toraldo di Francia, as we have noted previously.

It is important to note the sense in which we are using the expression 'not well defined'. We are supposing that there are four neutrons and that we may think of a collection with two of them. The problem is that we cannot offer a suitable way of choosing among the six possibilities we have. No one can, on the basis of the physics. The 'reality' seems to be hidden behind a veil,[994] so that the predicate *P* (here only roughly defined) 'to be a neutron of the nucleus of a $^7$Li atom which belongs to a collection with two of them' *has* an extension, although we cannot specify it precisely, say, by ostension. Let us recall that this represents a fundamental difference between quasi-sets and fuzzy sets; in short, in the case of the latter, the elements do have identity, but we do not know precisely *where* they are. Concerning quasi-sets, the borderlines of the collections are well defined in principle (actually, all we have is a cardinal—its quasi-cardinal), but the elements don't have identity. The nucleus of the sodium atom has (in principle) borderlines: either a neutron belongs to it or not, but the problem is that we cannot specify by ostension the neutrons themselves.

So, it is in this sense that we say that predicates like *P* do not have well-defined extensions: every qset of a certain class of similar qsets with the same quasi-cardinality may be considered as their extension as well.[995] Using a terminology which will be developed below, such predicates may be seen as *relations-in-intension*. Next we shall try to make sense of these claims from a formal point of view.

Let us call $S_\omega \mathcal{J}$ a higher-order modal logic (simple theory of types with modalities) described as follows. We shall begin by modifying the already defined set of types as follows. We shall let the set of *types* be the smallest collection $\Pi$ such that: (a) $e_1, e_2 \in \Pi$, and (b) if $\tau_1,\ldots,\tau_n \in \Pi$, then $\langle \tau_1,\ldots,\tau_n \rangle \in \Pi$. Here, $e_1$ and $e_2$ are taken to be the types of the *individuals*; the objects of type $e_1$ are again called *m*-objects and thought of as representing quanta. Once again following Schrödinger, we will suppose that the concept of identity

[994] Although we shall not explore it here, we note that this idea resembles a point made by D'Espagnat 1983, Chapter 9.

[995] We recall that, intuitively speaking, $Q$-similar quasi-sets have the same quantity of elements of the same sort.

cannot be applied to them. The language of the system $S_\omega \mathscr{J}$ contains the usual connectives (we suppose that $\neg$ and $\rightarrow$ are the primitive ones, while the others are defined as usual), the symbol of equality $=$, auxiliary symbols and quantifiers ($\forall$ is the primitive and $\exists$ is defined in the standard way) and the necessity operator $\Box$. With respect to variables and constants, for each type $\tau \in \Pi$ there exists a denumerably infinite collection of variables $X_1^\tau, X_2^\tau, \ldots$ of type $\tau$ and a (possibly empty) set of constants $A_1^\tau, A_2^\tau, \ldots$ of that type; we use $X^\tau$, $Y^\tau$, $C^\tau$ and $D^\tau$ perhaps with subscripts as meta-variables for variables and constants of type $\tau$ respectively.

The *terms* of type $\tau$ are the variables and the constants of that type; so, we have in particular individual terms of type $e_1$ and individual terms of type $e_2$. We use $U^\tau$, $V^\tau$, perhaps with subscripts, as syntactic variables for terms of type $\tau$. The atomic formulas are defined in the usual way: if $U^\tau$ is a term of type $\tau = \langle \tau_1, \ldots, \tau_n \rangle$ and $U^\tau_1, \ldots, U^\tau_n$ are terms of types $\tau_1, \ldots, \tau_n$ respectively, then $U(U^\tau_1, \ldots, U^\tau_n)$ is an atomic formula; so is $U^\tau = V^\tau$ if $\tau$ is not of type $e_1$ (the formal details can be completed without difficulty). Hence, once again, the language does not permit us to talk either about the identity or the diversity of the individuals of type $e_1$. The other formulas are defined as usual. A formula containing at least $U^\tau_1, \ldots, U^\tau_n$ as free variables sometimes shall be written $F(U^\tau_1, \ldots, U^\tau_n)$.

# 8.6 GENERALIZED QUASI-SET SEMANTICS

All that follows is developed within the quasi-set theory $\mathfrak{Q}$ presented in the previous chapter. When we speak about *sets, mappings* and other concepts which resemble the standard set theoretical ones, they should be understood as defined in the 'standard part' of quasi-set theory, that is, in the 'copy' of ZFU we can define within $\mathfrak{Q}$. Let $\mathbf{D} = \langle \mathbf{m}, \mathbf{M} \rangle$ **be an ordered pair where** $\mathbf{m} \neq \varnothing$ is a finite pure qset (that is, a finite quasi-set which has only *m*-atoms as elements) and $\mathbf{M} \neq \varnothing$ is a set (in the sense that it satisfies the predicate $Z$ of the language of $\mathfrak{Q}$). Furthermore, we suppose that $I$ is a non-empty set (whose elements are called an *index* or *state of affairs*).[996]

[996] Adapting Montague's approach to intensional logic, we may suppose that $I$ is the Cartesian product $W \times T$ where $W$ is a (quasi-)set of possible worlds and $T$ is a totally ordered set of instants of time. See Gochet and Thayse 1989, Chapter 2.

By a *frame* for $S_\omega \mathcal{J}$ based on $D$ and $I$ we mean an indexed family of qsets $(\mathcal{M}_\tau)_{\tau\in\Pi}$ where:

(i) $\mathcal{M}_{e_1} = {}_E\mathbf{m}$
(ii) $\mathcal{M}_{e_2} = {}_E\mathbf{M}$
(iii) For each $\tau = \langle \tau_1, \ldots, \tau_n \rangle \in \Pi$, $\mathcal{M}_\tau$ is a non-empty subqset of $[\mathcal{P}(\mathcal{M}_{\tau_1} \times \ldots \times \mathcal{M}_{\tau_n})]^I$.

If the equality holds in (iii), then the frame is *standard*. By a *general model (g-model* for short) for $S_{\omega\mathcal{J}}$ based on $\mathbf{D}$ and $I$ we understand an ordered pair

$$\mathcal{A} = {}_E\langle (\mathcal{M}_\tau)_{\tau\in\Pi}, \phi \rangle \tag{8.6.1}$$

such that:

(i) $(\mathcal{M}_\tau)_{\tau\in\Pi}$ is a frame for $S_{\omega\mathcal{J}}$ based on $\mathbf{D}$ and $I$
(ii) $\varphi$ is a quasi-function which assigns to each constant $C^\tau$ an element of $\mathcal{M}_\tau$. Then, in particular $\varphi(C^e_1) \in \mathbf{m}$ and $\varphi(C^e_2) \in \mathbf{M}$.

A *standard model* for $S_\omega \mathcal{J}$ is a g-model whose frame is standard.[997] Before introducing other semantic concepts, let us consider some examples which illustrate the peculiarities of such a semantics, as mentioned in the beginning of this section. The first two examples below show that the classical intensional case remains valid when the entities are not of the type $e_1$. The last two exemplify the specific case of Schrödinger logics, and are in conformity with some of Dalla Chiara's considerations mentioned in Chapter 5.

**Example 1.** *Let us consider a constant* $C^e_2$. *Since* $\mathcal{M}_{e_2} = {}_E\mathbf{M}$, *then* $\varphi(C^e_2) \in \mathbf{M}$, *as remarked above. This intuitively means that a constant of type* $e_2$ *'names' an element of a standard set (that is,* **M***is a set, according to the above definition of frame). This is not surprising, of course, since the given constant is a 'classical' one.*

**Example 2.** *Now we shall consider a constant* $C^{\langle e \rangle}_2$ *of type* $\langle e_2 \rangle$, *which stands for a 'property' of entities of type* $e_2$. *In this case, according to the above definition,*

$$\mathcal{M}_{\langle e_2 \rangle} \subseteq [\mathcal{P}(\mathcal{M}_{e_2})]^I = {}_E[\mathcal{P}(\mathbf{M})]^I.$$

*Then,* $\mathcal{M}_{\langle e_2 \rangle}$ *is a class of (quasi-)functions from I to* $\mathfrak{p}(\mathbf{M})$, *also according to the classical case. This is also not surprising, since the chosen constant is also 'classical'.*

[997] These definitions are obvious adaptations in the language of quasi-sets of those presented by Gallin (1975).

The next two examples will give us a better idea of intensions, as previously discussed.

**Example 3.** *Let us take a constant $C^{e}_{1}$. In this case, $\mathcal{M}_{e_1} = \mathbb{E}^{\mathbf{m}}$, and then $\varphi(C^{e}_{1}) \in m$, that is to say, that constant 'names' an m-atom. Since the m-atoms can be supposed to be indistinguishable, they cannot be individuated, counted etc., and so the denotation of $C^{e}_{1}$ must be ambiguous. We can say that a constant of type $e_1$ plays the role of a generalized noun, or g-noun for short.*

The commitment to quasi-set theory should be clear. It is precisely by considering such a mathematical framework that we can express the idea that a certain constant of the language does not represent a name of a well defined object of the domain. As in the case of electrons, it acts as a kind of *ambiguous* or *sortal* constant, since the entities to which it refers cannot be identified without ambiguity. So, by a *sortal constant*, or a *g-noun*, we mean a constant which refers (ambiguously) to an element of a certain class of indistinguishable objects, or objects given as *sorts* of a certain kind. We will pursue this idea below.

**Example 4**. *We now consider a constant $C^{\langle e \rangle}_{1}$ of type $\langle e_1 \rangle$, which stands for a 'property' of entities of type $e_1$. In this case, according to the above definition,*

$$\mathcal{M}_{\langle e_1 \rangle} \subseteq [\mathcal{P}(\mathcal{M}_{e_1})]^{I} = \mathbb{E}[\mathcal{P}(\mathbf{m})]^{I}.$$

*Then, $\mathcal{M}_{\langle e_1 \rangle}$ is a class of (quasi-)functions from I to $\mathfrak{p}(\mathbf{m})$. If $\mathbf{m}$ is a pure qset whose elements are all indistinguishable from one another, then the denotation quasi-function does not distinguish between qsets in $\mathfrak{p}(\mathbf{m})$, in the sense that any element of a class of Q-similar qsets acts as the 'extension' of the predicate $C^{\langle e \rangle}_{1}$ as well. This interpretation accommodates the intuitive idea that a predicate such as 'to have spin up in the x direction' does not have a well defined extension, as required by Dalla Chiara and Toraldo di Francia's analysis.*

This last example shows how to consider *relations-in-intension* of sort $U^{\langle e \rangle}_{1}$ within our formalism. This should not be taken to suggest that we are strongly committed to intensional issues only. As we hope to have made clear, our predicates *do have* extensions, but they are not well defined in the sense outlined earlier. Since the *m*-atoms have no proper names, the terms of type $e_1$ have no

precise denotation; they refer ambiguously to an arbitrary element of a certain class of the domain, which may be characterized by the particular chosen constant. We may properly say that such constants do not represent anything in particular: they lack a (precise, well-defined) referent, although they have a sense as constants, namely, the sense ascribed by the kind (sort) of entities they stand for (neutrons, say). The same holds for constants of type $\langle e_1 \rangle$ and for whatever constant of type $\tau = \langle \tau_1, \ldots, \tau_n \rangle$ where at least one of the $\tau_i$ is obtained (recursively) from $e_1$. As we have said, the relationship with sortal logics will be mentioned later.

Coming back to the formal details, we consider as the set of all *assignments over* a g-model $\mathscr{A}$, denoted by As($\mathscr{A}$), the set of all q-functions $f$ on the set of variables of $S_\omega\, \mathscr{J}$ such that $f(X^\tau) \in \mathscr{M}_\tau$, for every variable $X^\tau$ of type $\tau$. For any $f \in As(\mathscr{A})$, we denote by $f$ the extension of $f$ to the set of all constants, defined by

$$\bar{f}(C^\tau) = {}_E\rho(C^\tau) \in \mathscr{M}_\tau.$$

*If* $i \in I$ and $f \in As(\mathscr{A})$, then the notion of *satisfaction*, in symbols,

$$\mathscr{A}, i, f \,\mathrm{sat}\, A \tag{8.6.2}$$

is defined by recursion on the length of the formula $A$ as follows:

(i) $\mathscr{A}, i, f\mathrm{sat}U^\tau(U^\tau_1, \ldots, U^\tau_n)$ iff $\langle f(U^\tau_1), \ldots, f(U^\tau_n)\rangle \in f(U^\tau)(i)$

(ii) $\mathscr{A}, i, f\mathrm{sat}U^\tau = V^\tau$ iff $\langle f(U^\tau), f(V^\tau)\rangle \in \Delta_{\equiv}(\tau)$ where $\Delta_{\equiv}(\tau)$ is the 'pseudo-diagonal' of $\mathscr{M}_\tau$, which may be defined in quasi-set theory as the subqset of $\mathscr{M}_\tau \times \mathscr{M}_\tau$ whose elements are indistinguishable from one another (when $\tau \neq e_1$, this qset is the diagonal of $\mathscr{M}_\tau$ in the standard sense).

(iii) $\mathscr{A}, i, f\mathrm{sat}\ \Box A$ iff $\mathscr{A}, j, f\mathrm{sat}A$ for every $j \in I$

(iv) The usual clauses for $\neg$, $\rightarrow$ and $\forall$

A formula $A$ is *true* in a g-model $\mathscr{A}$ (denoted $\vDash_{\mathscr{A}} A$) iff $\mathscr{A}, i, f\mathrm{sat}A$ for every $i \in I$ and $f \in As(\mathscr{A})$. A set $\sigma$ of formulas of $S_{\omega\mathscr{J}}$ is *g-satisfiable* in $S_\omega\, \mathscr{J}$ iff for some g-model $\mathscr{A}$, index $i$ and assignment $f$, $\mathscr{A}, i, f\mathrm{sat}A$ for all $A \in \sigma$. A formula $A$ is a *g-semantic consequence* of a set $\Gamma$ of formulas, and we write $\Gamma \vDash_g A$, iff $\mathscr{A}, i, f\mathrm{sat}A$ for $i \in I$, $f \in As(\mathscr{A})$ and g-model $\mathscr{A}$ whenever $\mathscr{A}, i, f\mathrm{sat}B$ for every formula $B \in \Gamma$. If $\Gamma = \varnothing$, we write $\vDash_g A$ and say that $A$ is *g-valid* in $S_{\omega\mathscr{J}}$.

In the next section we present an axiom system for $S_\omega \mathcal{J}$ and prove a generalized completeness theorem for this logic.

## 8.6.1 The Theory $S_{\omega\mathcal{J}}$

The postulates of $S_\omega \mathcal{J}$ (axiom scheme and inference rules) are as follows:

(A1) $A$, where $A$ comes from a tautology in $\neg$ and $\rightarrow$ by uniform substitution of formulas of $S_\omega \mathcal{J}$ for the variables.

(A2) $\forall X^\tau(A \rightarrow B) \rightarrow (A \rightarrow \forall X^\tau B)$, where $X^\tau$ does not occur free in $A$.

(A3) $\forall X^\tau A(X^\tau) \rightarrow A(U^\tau)$ where $U^\tau$ is a term free for $X^\tau$ in $A(X^\tau)$ and of the same type of $X^\tau$.

(A4) $X^e_2 = X^e_2$

(A5) $X^e_2 = Y^e_2 \rightarrow \Box(X^e_2 = Y^e_2$

(A6) $\Box(U^\tau = V^\tau) \rightarrow (A(U^\tau) \rightarrow A(V^\tau))$, where $U^\tau$ and $V^\tau$ are free for $X^\tau$ in $A(X^\tau)$.

(A5) $\Box A \rightarrow A$

(A6) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(A7) $\blacklozenge A \rightarrow \Box\blacklozenge A$

(R1) From $A$ and $A \rightarrow B$ to infer $B$

(R2) From $A$ to infer $\forall X^\tau A$

(R3) From $A$ to infer $\Box$A

The usual syntactic notions are defined in the standard way, such as the concept of deduction ($\vdash$), formal theorem of $S_\omega \mathcal{J}$, the concept of consequence of a set of formulas, and so on. A set $\sigma$ of formulas is *consistent* if and only if there exists some formula of the language which is not derivable from $\sigma$ in $S_{\omega\mathcal{J}}$.

## 8.6.2 Soundness and Generalized Completeness

The Soundness Theorem for $S_\omega \mathcal{J}$ is formulated as follows. If $\vdash A$ in $S_\omega \mathcal{J}$, then $\models_g A$ in $S_\omega \mathcal{J}$. This result implies that if $\Gamma \vdash$ A, then $\Gamma \models_g A$ and that if a set of formulas $\sigma$ is g-satisfiable in $S_\omega \mathcal{J}$, then $\sigma$ is consistent.

The proof is obtained by showing that all the axioms of $S_\omega \mathcal{J}$ are g-valid and that the inference rules preserve g-validity. This follows from the fact that if

$\mathcal{A}$ is a g-model for $S_\omega \mathcal{J}$ and the term $U^\tau$ is free for the variable $X^\tau$ in $A(X^\tau)$, then for every $i \in I$ and $f \in As(\mathcal{A})$, it follows[998] that

$$\mathcal{A};i;f,\overline{f}\,(C^\tau)\,\mathrm{sat}\,A(X^\tau)\,\mathrm{iff}\,\mathcal{A},i,f\,\mathrm{sat}\,A(C^\tau) \tag{8.6.3}$$

where the terminology has an obvious meaning.

The generalized completeness theorem for $S_\omega \mathcal{J}$ is the converse of the above result; it is sufficient to prove that σ is consistent iff σ is g-satisfiable. The implication from right to left is straightforward, so we shall consider only the implication from left to right.

To begin with, let us assume that the consistent set σ omits infinitely many variables of each type, that is, there are infinitely many variables of each type which do not occur in any formula of σ. Then there exists[999] a sequence $\overline{\Sigma} = (\overline{\Sigma}_i)_{i\in\omega}$ of sets of formulas such that:

(i) $\Sigma \subseteq \overline{\Sigma}_0$
(ii) For each $i \in \omega$, $\overline{\Sigma}_i$ is a maximal consistent set of formulas in $S_{\omega\mathcal{J}}$
(iii) For each $i \in \omega$ and each formula $B(X^\tau)$, $\exists X^\tau B(X^\tau) \in \overline{\Sigma}_i$ iff $B(Y^\tau) \in \overline{\Sigma}_i$ for some variable $Y^\tau$ which is free for $X^\tau$ in $B(X^\tau)$.
(iv) For each $i \in \omega$ and each formula $B$, we have $\Diamond B \in \overline{\Sigma}_i$ iff $B \in \overline{\Sigma}_j$ for some $j \in \omega$.
(v) For each $i \in \omega$ and each formula $B(X^\tau)$, we have $\forall X^\tau B(X^\tau) \in \overline{\Sigma}_i$ iff $B(Y^\tau) \in \overline{\Sigma}_i$ for every variable $Y^\tau$ which is free for $X^\tau$ in $B(X^\tau)$.
(vi) For each $i \in \omega$ and each formula $B$, we have $\Box B \in \overline{\Sigma}_i$ iff $B \in \overline{\Sigma}_j$ for every $j \in \omega$.

The g-model relative to which the formulas of σ are g-satisfiable can be described as follows. First, we consider an equivalence relation on the collection $Tr_\tau$ of terms of $S_{\omega\mathcal{J}}$ of type $\tau$ such that $U^\tau$ is equivalent to $V^\tau$ if and only if $\Box(U^\tau = V^\tau) \in \overline{\Sigma}_i$ if $\tau \neq e_1$ and, if $\tau = e_1$, then $U^{e}_1$ is equivalent to $V^{e}_1$ (in this case we write $U^{e}_1 \equiv V^{e}_1$) iff for every formula $F$ that belongs to $\overline{\Sigma}_i$, it follows that $F[U^{e}_1/V^{e}_1]$ also belongs to $\overline{\Sigma}_i$. In other words, $U^{e}_1$ is equivalent to $V^{e}_1$ iff $U^{e}_1$ and $V^{e}_1$ can be replaced by each other in all their occurrences in any predicate in such a way that the resulting formulas are necessarily equivalent.

[998] As in Gallin op. cit.

[999] The existence of such a sequence can be proved by adapting the method presented in Gallin ibid.

The defined relation does not depend on $i \in \omega$. Then, by recursion on the type $\tau$, we define a set $\mathcal{M}_\tau$ and a mapping $\varphi_\tau$ from the set of terms of type $\tau$ into $\mathcal{M}_\tau$ such that:

(i) $\varphi_\tau$ is onto $\mathcal{M}_\tau$
(ii) $\varphi_\tau(U^\tau) \equiv \varrho_\tau(V^\tau)$ iff $U^\tau \simeq V^\tau$

First, let $\mathcal{M}_{e_i}$ be the quotient set $Tr_{e_i}/\simeq (i = 1, 2)$, that is, $\mathbf{m} = {}_E Tr_{e_1}/\simeq$ and $\mathbf{M} = {}_E Tr_{e_2}/\simeq$, and $\phi_{e_i}(U^{e_i}) = E]U^{e_i} \cong$ (these are the equivalence classes of $U^e_i$ by the relation $\simeq$). Then, by supposing that $\mathcal{M}_{\tau_\tau}$ and $\phi_{\tau_\tau}$ have been defined for $\tau < n$, we define the mapping $\varphi_\tau$ from $Tr_\tau$ into $[\mathcal{P}(\mathcal{M}_{\tau_0} \times \ldots \times \mathcal{M}_{\tau_{n-1}})]^\omega$, where $\tau = \langle \tau_0, \ldots, \tau_{n-1} \rangle$, as follows:

$$\left\langle \phi_0(U_0^{\tau_0}), \ldots, \phi_{n-1}(U_{n-1}^{\tau_{n-1}}) \right\rangle \in \phi_\tau(U^\tau)(i) \tag{8.6.4}$$

if and only if the formula $U^\tau(U^\tau_0, \ldots, U^\tau_{n-1})$ belongs to $\overline{\Sigma}_i$. If we let $\mathcal{M}_\tau$ be the range of $\varphi_\tau$, then the conditions 1 and 2 above are met. The g-model based on $\mathbf{D} = {}_E\mathbf{m} \cup \mathbf{M}$ and index set $I = {}_E\omega$ is then the ordered pair $\mathcal{A} = \langle (\mathcal{M}_\tau)_{\tau \in \Pi}, \varphi \rangle$, where $\varphi(C^\tau) = \varphi_\tau(C^\tau)$ for every constant $C^\tau$. So, by induction on the length of the formula $A$, we may prove in the same way as presented by Gallin for the 'classical case', that

$$\mathcal{A}, i, \mu \text{ sat } A \text{ iff } A \in \overline{\Sigma}_i \tag{8.6.5}$$

for every $i \in I$, where $\mu \in \text{As}(\mathcal{M})$. In the case where $i = 0$ and $\mu = f$, we obtain the desired result.

## 8.6.3 Comprehension and Other Axioms

Our logic can be extended to a system which encompasses all the instances of the following *Comprehension Schema*, where $\tau = \langle \tau_1, \ldots, \tau_n \rangle$ is a type and $X^\tau$ is the first variable of type $\tau$ which does not occur free in the formula $F(X^\tau_1, \ldots, X^\tau_n)$:

$$\exists X^\tau \Box \forall X^{\tau_1} \ldots \forall X^{\tau_n} (X^\tau(X^{\tau_1} \ldots X^{\tau_n}) \leftrightarrow F(X^{\tau_1} \ldots X^{\tau_n})). \tag{8.6.6}$$

This schema, which is valid in all standard models of $S_\omega \mathcal{J}$, formalizes the principle that every formula $F(X^\tau_1, \ldots, X^\tau_n)$ with free variables determines a relation-in-intension (a predicate). In considering a g-model $\mathcal{A} = \langle (\mathcal{M}_\tau)_{\tau \in \Pi}, \varrho \rangle$ for $S_{\omega \mathcal{J}}$, if $U^\tau_1, \ldots, U^\tau_n$ are respectively elements of

$\mathcal{M}_{\tau_1},\ldots,\mathcal{M}_{\tau_n}$, the predicate $F$ defined by

$$F(i) =_{\text{df}} \{(X^{\tau_1}\ldots X^{\tau_n}) : \mathcal{A}; i; f, U^{\tau_1},\ldots,U^{\tau_n} \text{sat} F(X^{\tau_1}\ldots X^{\tau_n})\} \tag{8.6.7}$$

for all $i \in I$ and assignment $f \in As(\mathcal{A})$ belongs to $\mathcal{M}_\tau$. Consequently, the g-model is also a g-model for $S_\omega \mathcal{J}$ plus the Comprehension Schema, and the completeness theorem is also true for this extended logic.

The *Principle of Extensional Comprehension*, which says that every formula with free variables determines an (extensional) *n*-ary relation can also be formulated in the language of $S_\omega \mathcal{J}$, as can the axioms of infinity and choice, although they are not important to us here.

As is the case in Gallin's intensional system, the Principle of Extensional Comprehension can be proved to be independent of the axiomatics of $S_\omega \mathcal{J}$ plus the Comprehension Schema. That is, there are g-models of $S_\omega \mathcal{J}$ plus Comprehension in which the extensional comprehension principle fails.

## 8.6.4 General Discussion

The system $S_\omega \mathcal{J}$ and its corresponding quasi-set semantics show that it is possible to treat formally the idea that there may exist predicates which don't have well-defined extensions, that is, predicates which can be regarded as well-defined (let us call them 'sharp' predicates) but such that their extensions cannot be determined without a certain ambiguity. We recall our previous example: suppose the predicate $P$ is defined so that $P(x)$ says that $x$ is a sub-collection of neutrons of the nucleus of a $^7$Li atom which has two elements. Of course there is a sense in forming such a predicate and, further, the predicate is not something 'vague' as the predicate 'to be bald' is, for physicists know perfectly well what it means and how to specify whether a certain physical object has such a property or not. In the case of 'vague' predicates, like 'red', 'bald' or 'intelligent', as discussed in the philosophical literature, the predicates are not perfectly well defined for, given a certain man, say, we may be in doubt whether he is bald or not. With respect to the predicate $P$ above, the situation is of course different, for the vagueness is not *in the predicate*, which is well-defined in physical contexts, but rather seems to be *in the world*, for the doubts

we may have concern the *entities* we are considering, as in the case of the precise determination of the 'two' neutrons that should belong to its extension.[1000]

So, once we know that there are four neutrons in such a nucleus, and since the neutrons are, in principle, absolutely indistinguishable, then there are six possibilities of forming a sub-collection with just two of them. The problem, as we have said, is that we cannot distinguish among them. As another example, think of the two electrons of the level 2*s* and the two electrons of the level 3*s* of a silicon atom (its atomic number is 14, and the electronic decomposition is $1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^2$). Taking into account the Indistinguishability Postulate, there are in fact precisely six possibilities of considering sub-collections with two elements which differ *solo numero* from a collection with four of them (there are 15 possibilities if the collection has quasi-cardinal 6). This is what we would expect from quantum statistics, as we have seen before.

Example 4 above suggests that this situation can be described within our system, unlike the case of the semantics based on standard set theories.

# 8.7 QUANTUM SORTAL PREDICATION

In this section, we consider a possible way of applying the general idea concerning sortal predicates to the quantum realm. Roughly speaking, sortal predicates involve a way of counting things that fall under them, and it is usually accepted that there exists a criterion of identity for such things. Here we discuss in what sense terms like 'electron', 'proton' and others suggested by quantum physics can define 'quantum-sortal predicates', but such that neither identity criteria nor counting processes can be associated with the objects that fall under them. A formal analysis of such quantum-sortal predicates is outlined, and its semantic aspects dealt with by means of quasi-set theory. We shall begin by recalling some basic features of sortal predication.

## 8.7.1 Sortal Predication

The difficulties in providing a clear distinction between *sortal predicates*, which are associated with terms like 'man', 'tree' and 'book' and other predicates,

[1000] This idea—that quantum entities are an example of vagueness 'in the world'—is further explored in French and Krause 2003. It is our suspicion that claims that vagueness must necessarily be a matter of *language* represent a failure to appreciate the nature of quantum mechanics, together with an implicit commitment to standard set-theory and classical logic.

which arise from terms like 'green' and 'thing', have been acknowledged in the philosophical literature. The subject is related to a more general discussion concerning the nature of general terms and has its roots (at least) in Aristotle's concept of second substance. We shall not delve into the details of this discussion here, except to note that according to Strawson,[1001] general terms are divided up into *characterizing* and *sortal* terms, while Geach talked in terms of *adjectival* and *substantival* general terms respectively.[1002] Lowe, in his analysis of the issue, has chosen *adjectival* and *sortal* respectively to designate these terms, and cites Dummett's criteria for distinguishing them.[1003] The latter may be summarized as follows:

(i) Adjectival terms have associated with them a *criterion of application*, and by this we understand a general principle that determines to which *individuals* the considered term correctly applies.
(ii) In addition to a criterion of application, a sortal term also has a *criterion of identity* associated with it. This is a principle that determines the conditions under which one individual to which the term applies can be said to be the *same* as or *distinct* from another.

As Wallace notes, the distinction between adjectival and sortal terms is quite subtle, and it is difficult to find a clear way of providing an 'objective' understanding of these criteria. Nevertheless the criterion of identity has been associated with a *counting process*, that is, something which should enable us to *count* things that fall under a certain sortal term. Take for instance, the term 'tree' (this is Lowe's example). Of course we may suppose that we have a criterion of application which enables us to apply the term to some *given* object; furthermore, we can *count* trees, at least in principle. So, 'tree' is a sortal concept, and 'to be a tree' is a sortal predicate. But 'green' (another of Lowe's examples) is not, for while we have a criterion of application for identifying green things, despite the vagueness associated with this predicate (we shall return to this point below), we don't have an associated counting process for green things. In trying to count green things in a wood, we would not know what things to count; should we count only the trees in a wood as green things? If the grass is also to be counted, then should the grass be counted as just one object or should we count also every leaf of grass? The same holds for the leaves of the trees. And what about the parts of the leaves?

[1001] Strawson 1959, p. 168.

[1002] Geach 1962.

[1003] Lowe 1995, Chapter 5. See also Lowe 1994.

Since a leaf can be divided up, say, into circular or triangular parts, are these parts also to be counted as distinct green things? Similarly, the same holds for terms like 'thing'; the reader may try to count the things there are in the room now. How many are there? Are the parts of the things (say, the lock of a door) also 'things' which should be added to the list? And what about the components of the lock? All of this exemplifies, albeit crudely, the claim that 'green' and 'thing' are examples of non-sortal terms.

The role they play in the counting process is generally acknowledged to be the distinctive feature of sortal terms, and of sortal predicates in particular. Thus, Wallace says: "a sortal predicate '*F*' provides a criterion for counting things that are '*F*' ", and continues: "[i]f '*F*' is a sortal predicate, you can find out how many *F*'s there are in such and such a space by counting".[1004] The interesting thing is that the discussion is typically developed in connection with, say, 'macroscopic' objects; that is, the efforts are directed at describing a *given thing* as falling under either a sortal or an adjectival predicate and the examples concern 'trees', 'mountains', 'green things', 'bald men' and so on, that is, *individuals* of a sort (recall Dummett's criteria above). This is of course not so strange for, as Auyang notes, "the paradigms of objects are the things we handle everyday".[1005]

Furthermore, even among sortal terms there are certain distinctions to be taken into account. For example, as Lowe notes, sortal terms like 'green' and 'mountain' have different criteria of identity. Let us follow Lowe for a moment in order to see what is involved; he says:

> The criterion of identity for *trees*, for instance, is very different from the criterion of identity for *mountains* (…) *Trees* (…) can undergo very considerable changes of shape and position [he is referring to the possibility for a tree we know very well to be transplanted to another place in our garden during our absence] while remaining numerically the same, that is, while persisting identically through time. By contrast, it does not make much sense to talk of *mountains* undergoing radical changes of shape and position (…) If the land falls in one place and rises in another, we do not say that a mountain has *moved*, but rather that the mountain has ceased to exist and another has been created. (To be sure, we do allow 'small' changes in the shapes and positions of mountains, and this does potentially lay us open to paradox, since a long series of small changes can add up to a large change—as in the notorious paradox of the bald man. This, however, just shows that 'mountain', like many other general terms in ordinary language—such as 'red' and, indeed, 'bald'—is a *vague* term).[1006]

[1004] Wallace 1965.

[1005] Auyang 1995, p. 11.

[1006] Lowe 1995, pp. 95–6.

This criterion of remaining the same through time concerns the issue of trans-temporal identity, as we remarked earlier,[1007] but it will not be discussed here. Although Lowe does not discuss vague terms, the quotation above suggests that among sortal predicates (like 'to be bald') there are also those which are *vague*, and of course they also deserve attention.

This discussion in the literature has taken the topic in several interesting and important directions. Nevertheless, little, if anything, has been done to relate it to aspects of the epistemology of quantum physics. We shall present a framework for such a possible relationship. As a corollary, we shall discuss in what sense the considerations given here might contribute to the development of so-called sortal logics (the logic of sortal predication). Thus, consider general terms like 'electron', 'proton', 'strings' and other terms provided by physics. Are these terms merely adjectival or are they sortal terms? In the second case, do these objects have a criterion of counting despite their indistinguishability? This is what we shall discuss in the next section. In doing so, we shall also touch on the case of vague sortal terms.

## 8.7.2 Quantum-Sortal Predicates

Let us consider the predicate 'to be a proton of a 7-Lithium atom', which we shall call *Pr*. Alternatively, we could consider the following predicate (which we could use to communicate a certain experience to our audience): '(to be) the atom which was ionized negatively by capturing an electron and which, a short time later, has reverted to a neutral state by releasing an electron'.[1008]

These cases suggest that, first of all, general terms like 'proton', 'electron', etc. are not mere adjectival general terms (in the above sense), since they do not merely *characterize* (to use Strawson's words) an object as a such-an-such.[1009] Perhaps we could say, in Quine's sense, that they also do not divide their reference: a proton cannot be divided up into smaller parts, and the same holds for other kinds of particles.[1010] Terms like these apparently refer to certain kinds (sorts) of entities which are closer to *trees* in a wood or to *dogs* in a city than to *green* things in a wood or *things* in a room. Actually, we may talk of the three protons in the nucleus of the Lithium atom, or of its four

[1007] See also Quinton 1973, pp. 58ff.

[1008] This example is taken from Lowe 1994.

[1009] Strawson op. cit., p. 167.

[1010] Even if we consider annihilation/creation processes, we have to accept the fact that, within a certain context, for instance when speaking of the electronic aspects of a certain atom, a proton remains a proton.

neutrons without confusion of concepts. In other words, terms like 'electron', 'neutron', 'proton' and so on should be treated as sortal terms of a kind. But of what kind? Do they have the aforementioned characteristics attributed to sortal terms? It turns out they do not, for reasons which should be clear given the theme of this work.

Our previous discussions suggest that terms like 'proton' and 'electron' have associated with them a criterion of application, although, on the Received View, at least, we don't have the 'individuals' to properly apply this criterion. In Toraldo di Francia's sense, as discussed previously, they are considered to be *nomological*, that is, they *come with the theory*. However, these terms are not merely adjectival. So, let us call such predicates *quantum-sortal predicates* (q-s-predicates). Then we can say the following:

(i) Quantum-sortal predicates have a criterion of applicability which tells us to what kind of entity they apply. For instance, the predicate *Pr* above applies to protons, not to electrons, and the distinction between these two categories of quantum entities may be assumed to be no less clear to the physicist than the terms 'tree' or 'mountain' are to the average person, for physicists have the possibility of recognizing (either by theoretical or by experimental means), whether a given physical system is, say, an electron system or not.

(ii) Although we cannot say that there is a well-defined criterion of identity which enables us to distinguish between 'two' objects that fall under a certain 'q-s-concept', as in the case of the three protons of the Lithium atom, even so we usually refer to a *quantity* of them: "the objects of physics are associated with natural numbers".[1011]

This associated *number*, although obtained in different ways, such as for instance by means of Feynman diagrams, is of course not given by *counting*, if by this we understand the usual attribution of an ordinal to their collection. So, we have the situation where a certain collection of 'objects', roughly speaking, may have a cardinal, but not an ordinal. These collections have already been identified with *quasi-sets*.

To provide a further characterization of the predicates we are introducing, let us make a comparison with the standard examples. Thus, following Terricabras

[1011] Toraldo di Francia op. cit., p. 306. As we have noted, he also considers the case of virtual particles, when even the cardinal of a collection of them may not be defined, but we shall keep to the case where the 'number' is taken into account.

and Trillas,[1012] let us call *Fregean* a predicate $F$ which induces a bipartition in the domain of discourse into two disjointed subsets whose union gives again the whole domain. In other words, a Fregean predicate $F$ enables us to define a mapping $f: D \rightarrow (0,1)$, where $D$ is the domain, such that:

$$D = f^{-1}(0) \cup f^{-1}(1),$$

and these two sets are disjoint. In the standard semantics, the set $f^{-1}(1)$ is the *extension* of the predicate $F$, while $f^{-1}(0)$ is its complement relative to $D$. So, given any $x \in D$, one of the two possibilities holds: either $F(x)$ is true (when $x \in f^{-1}(1)$) or $\neg F(x)$ is true (when $x \in f^{-1}(0)$).

*Vague* predicates may be characterized as follows: a vague predicate $V$ induces a mapping $v: D \rightarrow [0,1]$ (the closed interval of real numbers) such that

$$D = v^{-1}(0) \cup v^{-1}(1) \cup \cup_{r \in (0,1)} v^{-1}(r),$$

where $\cup_{r \in (0,1)} v^{-1}(r)$ is a non-empty set and these three sets are pair-wise disjointed. Of course the case of Fregean predicates can be incorporated within this framework by supposing that in this case this last set is empty. So, if $\cup_{r \in (0,1)} v^{-1}(r)$ is not empty, we can say that there are objects in $D$ for which we cannot assert either that they have the property $V$ or that they do not. For instance, if $V(x)$ means '$x$ is bald', then if $x \in v^{-1}(0)$, we say that $x$ is not bald; if $x \in v^{-1}(0)$, we say that $x$ is bald, but if $x \in \cup_{r \in (0,1)} v^{-1}(r)$, we should say that $x$ is 'more or less' bald, depending on the place of the $r$ in the interval [0,1]; the closer $r$ is to 1, the more $x$ is bald. Clearly, as noted by Terricabras and Trillas, a semantic analysis of vague predicates could be achieved using fuzzy sets.

However, it should be noted that in both the cases considered (Fregean and vague predicates) we are dealing with *individuals*; the worst case (involving vague predicates) is the situation where we have a certain 'well-defined' individual, say the well known Mr. X, but we are in doubt whether he is bald or not. The uncertainty is *epistemological* only. A different situation is posed by quantum objects, where we are faced with a different kind of uncertainty: an *ontological* one. The problem now is not with the predicates themselves: they are not vague at all, for the physicist knows very well what a certain object must satisfy, say, to be classified as a proton. She *knows* protons theoretically

[1012] Terricabras and Trillas 1989.

and experimentally. The uncertainty concerns the entity itself. In other words, with indistinguishable quantum objects the problem is to identify the extension of a predicate like 'to be a proton of a $^{7}Li$ atom', for any collection with three protons suffices, and this is quite different from, say, 'to be the Home Secretary of the UK' (put another way, the extension of this last predicate obeys the Axiom of Extensionality of set theory, while the extension of the former does not). Thus, in the semantic analysis of such kinds of predicates, even fuzzy sets are not of much help, for while they enable us to deal with epistemological uncertainty in the above sense, they do not help us in dealing with ontological uncertainty.

So, we may say that the *quantum-sortal predicates* (for the lack of a better term), have the following main characteristics:[1013]

(i) They have a criterion of applicability in Dummett's sense mentioned above.
(ii) Instead of a criterion of identity, there is a *number criterion*, a principle which enables us to say that in certain situations the predicate truly applies to a certain number (generally finite) of entities, yet sometimes there is no counting process associated with them. This number may be identified with the occupation numbers of quantum field theory,[1014] and may vary from one application to another.
(iii) In certain situations, such as those involving indistinguishable quantum entities, the extension of the predicate is not well defined, in the sense already explained that another collection of similar objects with the same cardinality may act as its extension as well. So, we may say that there is a kind of *opacity* involving at least some objects of the domain, for the issue becomes not that of predicates lacking 'sharp boundaries' (as in the situations involving vague predicates), but rather that of the objects to which the predicates apply lacking individuality (under the Received View).

It seems clear to us that such predicates have their place in the pantheon of general terms and that they should be considered in the semantic analysis of predication and reference in general. Hence, we need a way of characterizing them formally. This is what we shall do in the following sections.

[1013] In Krause and French 2005, these predicates were termed *opaque*; see below.

[1014] See Auyang op. cit., pp. 159–60.

## 8.7.3 Sortal Logics

Sortal predication has been treated formally by means of the related concept of relative identity. Geach suggested that there is no 'absolute' identity, and that all identity statements are relative. Thus, when we say that '$x$ is identical with $y$', we intend to say that '$x$ is the same $S$ as $y$', where $S$ is understood as a sortal predicate ("a count noun", according to Geach).[1015] This can be written as $x =_S y$, following Stevenson.[1016] The aim of sortal logics is to treat these predicates differently from standard one-placed predicates, for in the usual first-order logic we can write $x = y \wedge S(x)$ to mean '$x$ is the same $S$ as $y$'. But, in this case, what distinguishes $S$ as a sortal predicate in the sense already explained? The way this distinction is established has been a source of controversy.

First of all, it should be noted that there has not been a 'proliferation' of sortal (formal) systems in the literature. Pelletier's review provides a general account of the subject, and he mentions the works of Smiley, Wallace, Stevenson (op. cit.) and Tennant on sortal logic.[1017] Our arguments here do not depend on a revision of these systems.

Thus Pelletier insists that in trying to characterize a sortal logic, these systems have provided only 'syntactic sugar', for according to him none of them provides a clear distinction between sortal and standard one-placed predicates; as he puts it,

> … the accounts produced are merely notational variants of classical restricted quantification theory—which of course is a mere notational variant of classical quantification theory (…) In restricted quantification theory we 'abbreviate' formulas of the form $\forall x(Fx \rightarrow Gx)$ and $\exists x(Fx \,\&\, Gx)$ respectively as $(\forall x\colon Fx)Gx$ and $(\exists x\colon Fx)Gx$. The latter formulas appear to have the syntactic unit 'quantifier phrase' (if $F$ stood for 'dog', then the quantifier phrases would be 'every dog' and 'some dog'). But in restricted quantification theory this is *mere* appearance, for these formulas have precisely the same truth conditions as the original unrestricted formulas. Exactly the same formulas are theorems; exactly the same arguments are valid, after translation from one idiom to another (…) True sortal logic resides in restricted quantification theory exactly to the same extent that it resides in unrestricted quantification theory (…) and then one concludes that *none* of these alleged 'sortal logics' adequately represents the desired doctrine.

[1015] Geach 1967.

[1016] Stevenson 1975.

[1017] See Pelletier 1992, where the references are given.

Pelletier goes on to make an interesting remark: "[a] sortal concept is a (mental? objective?) concept of a *kind* or *sort* of individual. A sortal predicate is a linguistic item which is correlated with a sortal concept. According to this account there is no such thing as an individual *tout court*; instead, individuals come already pre-packaged as individuals-of-the-*F*-type (where *F* is a sortal concept)." Given what we have said of the characterization of quantum objects, under the Received View, we may be able to understand why there are no truly sortal logics, as Pelletier notes: all of them are committed to standard languages and, more importantly, to standard set theories in their semantic aspects. It seems that while physics has moved on from classical physics to quantum (and relativistic) physics, logic and mathematics still retain the languages which refer to individuals and collections of distinguishable objects (sets).

However, a solution can be presented, at least with regard to our quantum-sortal predicates; if we regard logic as involving also its semantic aspects, then a logic of such predicates can perhaps be achieved in the context of quasi-set theory. Thus, if the collections described by this theory are taken to be the extensions of certain predicates, then these predicates could legitimately be termed 'quantum sortals'. Furthermore, owing to the indistinguishability of the elements of these collections, any two with the same cardinal would be taken as the extension of the predicate, so vindicating the above requirement about quantum objects involving the Indistinguishability Postulate.

## 8.8 SEMANTICAL ANALYSIS

Hence the suggestion is that sortal-quantum predicates can be semantically characterized by using quasi-sets. As we have seen, any quasi-set belonging to a collection *C* of quasi-similar quasi-sets (recall that these are those quasi-sets which have the same quasi-cardinality and whose elements are related by the indistinguishability relation ≡) may act as the extension of predicates like Pr, defined above (where, we recall, *Pr*(*x*) stands for '*x* is a proton of a $^{7}$Li atom'). Thus, it makes sense to say that they do not have a well-defined extension, since all of these quasi-sets are indistinguishable by the weak extensionality axiom. Furthermore, owing to the characteristics of the *m*-atoms, the elements of their (ambiguous) extension cannot be regarded as 'individuals *tout court*', but can be characterized only in terms of the 'sort' delineated by the properties which (albeit ambiguously) define the relevant collection *C* (Dalla Chiara

and Toraldo di Francia identify the conjunction of these properties with the *intension* of the quasi-set).

A logical system that might be useful for characterizing quantum-sortal predicates is a slight variation of the system $S_{\omega}$ $\mathscr{J}$ presented in the previous section. By considering its quasi-set semantics, the predicates of type $C^{\langle e \rangle}_{1}$ may represent our quantum-sortal predicates. All that was said in the previous section helps to underpin this suggestion, for in this case (using the notation we presented in the previous section), $\mathfrak{F}_{\{e_1\}} = {}_{E}[C]^{I} \subseteq [\mathcal{P}(\mathfrak{F}_{e_1})]^{I}$. Then, $\rho(C^{\{e_1\}}) \in \mathfrak{F}_{\{e_1\}}$, that is to say, it is a (quasi-)function from *I* to *C* (which, let us recall, is a collection of quasi-similar quasi-sets). In other words, $\varrho(C^{\langle e \rangle}_{1})$ is a quasi-function from *I* to *C*. If *m* is a pure quasi-set whose elements are all indistinguishable from one another (that is, they stand in the relation ≡), then the denotation function does not distinguish between quasi-sets in *C*. In this case the only difference among the subquasi-sets of *m* concerns their cardinality; that is to say, if $\varrho(C^{\langle e \rangle}_{2})$ is $x$, this $x$ does not have a precise definition, for any quasi-set *y* such that $x$ and *y* are similar could act as the denotation of $C^{\langle e \rangle}_{2}$ as well.

In other words, these constants exemplify the case where a predicate does not have a well-defined extension (in the sense that every quasi-set of a certain class of similar quasi-sets may be considered as their extension); these predicates may not only be viewed as relations-in-intension of sort $U^{\langle e \rangle}_{1}$, as we have suggested, but also as quantum-sortal predicates, covering the situation where, for instance, a physicist is measuring a certain property of a quantum system, say the spin of a collection of electrons. Suppose that she has chosen the $x$ direction and has stated how many electrons have spin up and how many have spin down. However, she could choose another direction, say the $z$ axis, thereby obtaining collections of quantum states with the same cardinality, but for which one cannot say that these collections are the same or that they are distinct from the first. So, the predicate 'to have spin up' does not have a precise denotation, for any collection with the right number of electrons can act as its extension, and no counting process (in the mathematical sense of attributing to it an ordinal) can be applied. Let us conclude by agreeing with Toraldo di Francia that "the intepretation of the logical concept of extension may definitely need a profound revision in modern physics".[1018]

[1018] Toraldo di Francia op. cit., p. 306.

# 9
# The Logic of Quanta

> Are there quantum objects? What are they if they are neither given nor resemble anything familiar? To answer these questions we have to abstract from the substantive features of familiar things, delineate the pure logical forms by which we acknowledge objects and show how the forms are fulfilled in quantum theories. We have to explicate, in general terms and without resorting to the given, what we mean by objects. The clear criteria will enable us to affirm the objectivity of quantum theories.
>
> Auyang 1995, p. 5

Let us return to our discussion of the underdetermination of metaphysical packages. In Chapter 4, we mentioned attempts that have been made to break the underdetermination by appealing to the claim that the particles-as-non-individuals option meshes better with the framework of quantum field theory (QFT) and hence is to be preferred. Setting aside the problematic methodological issue concerning the nature and role of 'surplus structures', our aim in this chapter is to consider the metaphysical issues surrounding this claim. In particular we shall examine Teller's suggestion that QFT can be understood as positing non-individual quanta and we shall put forward quasi-set theory as a suitable formal framework for such entities. We shall begin with a short outline of the relevant issues in the foundations of QFT, to illustrate the similarities with and differences from standard quantum mechanics.

# 9.1 THE NATURE OF QFT

When it comes to the metaphysics of quantum fields, there is, of course, a whole plethora of issues to consider.[1019] Here we shall focus on just a very few, associated with the status of particles within QFT and the nature of quanta.

Typically, when discussing these issues, one begins by explaining how a quantum field is constructed.[1020] The reason for this initial move has to do precisely with the metaphysics of particles in standard, 'first quantized', quantum mechanics and how that metaphysics is incorporated into the foundations of QFT. Basically, one can obtain a quantum field either by taking a classical field and quantizing it—a process known as 'field quantization'— or by taking an assembly of indistinguishable (and non-interacting) quantum particles, described by the appropriate wave equation and quantizing that—a process termed, unsurprisingly, 'second quantization'. Since we shall be considering the metaphysical underpinnings of these quantum fields, it will be worth giving a brief sketch of how they are constructed.

Let us begin with the standard assumption that a classical field can be decomposed into the weighted sum of 'modes', each of fixed wavelength. It can be shown that the energy of the field can be decomposed in a similar manner and that each mode independently contributes to the total energy. There is an obvious analogy here with the case of the simple harmonic oscillator,[1021] which, of course, is straightforward to quantize. The field energy is then equal to the weighted sum of the energies of the separate modes (plus a constant, interpreted as the 'zero-point' energy, to which we shall return shortly), where the 'weight' is given by the excitation number of each mode. This latter number is then identified with the number of particles, or quanta, associated with that state of the field and what we appear to have arrived at is a particle interpretation of the quantized field. However, as Redhead notes, "… these 'particles' have no TI labels attached to them. They have no intrinsic individuality".[1022]

In other words, these quanta are not regarded as individuals from the very start; they are non-individuals and it is on this basis that it has been argued that the 'particles-as-non-individuals' package of first-quantized quantum mechanics 'meshes' with QFT.

[1019] For a useful introduction see Cao 1999 or Huggett 2000.

[1020] See Redhead 1988; Huggett op. cit.

[1021] Huggett op. cit., pp. 622–3.

[1022] Redhead op. cit., p. 14.

Before we move on to outline second quantization, let us consider this conclusion in just a little more detail. One might ask the simple question, how is it that we have ended up with 'particles' to which we cannot attach labels and which have no 'intrinsic individuality'? The equally simple answer is that we began with a field and its associated description for which there is nothing corresponding to particle labels. However, it might well be felt that this is too simple an answer. One might reasonably expect certain aspects of the metaphysics of the classical field to be effectively 'transmitted' through the construction process to its quantum equivalent. We recall from Chapter 2 that a classical field can be regarded either in terms of properties instantiated at a set of space-time points, or as substantival in its own right. In both cases the field properties—such as the energy—are quantized, of course, but one would not expect the physics—by its very nature—to touch on the underlying metaphysics. Thus, we still have an underlying continuum of space-time points (flat in the first instance—how one accommodates curved space-times within QFT is of course a further thorny problem) and it might be suggested that, just as in the classical case, a form of Spatio-Temporal Individuality (STI) could be introduced here. The obvious objection is that this would require the quanta to be *localizable*, which is a famously problematic issue and one which we shall touch on below; for the moment we simply wish to note that non-localizability arises as a consequence of the relevant physics and thus if this is given as the 'source' of the non-individuality, it has nothing to do with the lack of labels.

Alternatively, one could argue that the substantival nature of the classical field has been inherited by its quantum counterpart and again this could serve as the basis of a form of transcendental individuality (TI) for the quanta. The point is similar to that made in the case of first quantized particles: even though there are no labels explicitly introduced within QFT—not even introduced and then shuffled, as in the quantum mechanical situation—this does not prevent us from regarding the quanta as substantival and hence individuals, albeit 'strongly' indistinguishable ones. Again, one might recoil at the Scholastic flavour of this metaphysics (even more so in the QFT case, precisely because we don't even begin with labels which are then permuted) but nevertheless it could be sustained on philosophical grounds. Of course, there may be further aspects of the physics which are hard to reconcile with a particle picture and as we have indicated, we shall consider these shortly.

Before we do, let us sketch the process of second quantization. Here we start with a set of non-interacting indistinguishable particles described by

the *N*-particle wave equation (thus the first quantized aspect is two-fold, as represented by the use of both the Schrödinger equation and the consequences of quantum statistics). In this case, the energy is already quantized, of course, and is a weighted sum of the energies of the one-particle states. The structural similarity between this and the energy spectrum of a collection of independent oscillators[1023] suggests the treatment of the one-particle wave equation as if it were a real field and subjecting it to a *second* quantization.

Thus we have two ways of obtaining a quantum field and, as we have said, numerous issues arise as to whether, for instance, we have the same 'animal' in both cases, as Redhead puts it,[1024] how we should understand the differences between the treatments of bosons and fermions and so on.[1025] Again, we shall simply be concerned with the question of whether the supposed non-individuality of the resulting quanta is an inevitable result of this process. Obviously, if one were to regard the assembly with which one begins as a set of non-individuals, then it would be natural to understand the field quanta in the same terms. But what if one had adopted the alternative package, and initially understood the particles as individuals? Again, the fact that we have no particle labels within the QFT formalism is not grounds, on its own, for abandoning the metaphysics we started with.

Indeed, van Fraassen has argued that, just as in the case of quantum mechanics, one can maintain a form of metaphysical underdetermination within the framework of QFT, since "It is equivalent to a somewhat enriched and elegantly stated theory of [individual] particles. That we can take it as a description of a world that is particle-less only masquerades as an incompatible alternative".[1026] The basis of this claim is the relationship of representation which is purported to hold between models of QFT and 'concrete' constructions of Fock space carried out within de Muynck's 'labelled particle' approach.[1027] However, this claim has been criticized by Butterfield, who casts doubt on the empirical equivalence that is supposed to exist between QFT and many-particle quantum theory.[1028] In particular, he has suggested that the existence, within QFT, of states that are superpositions of particle number render this claim false.[1029] The idea here is that on any reasonable account of individuality, the number of individuals must be definite. To deny the latter, Butterfield argues,

[1023] Redhead op. cit., p. 14.

[1024] op. cit., p. 15.

[1025] For further details, again, see Redhead op. cit., Huggett op. cit. and Cao op. cit.

[1026] van Fraassen 1991, p. 436.

[1027] de Muynck 1975.

[1028] Butterfield 1993. de Muynck himself argued that the standard theory and his own formalism were, in fact, inequivalent.

[1029] See also Huggett op. cit., p. 624.

leaves one open to the objection that the entities concerned are not individuals at all. Of course, one might admit this objection but insist that all it shows is that we have to give up the notion of *individual* particles. Thus Redhead has attempted to construct an 'extended' particle interpretation which can allow for superpositions of states with different particle numbers.[1030] However, he has since abandoned this attempt in the light of further problems which afflict the particle framework.[1031] Let us take a quick look at these problems before returning to the issue of accommodating non-individual quanta.

To begin with, it seems plausible to suppose that the existence of particles as individuals should not be dependent on the motion of the observer. However, that is what the famous Unruh effect appears to imply: even when the quantum field is in its vacuum state, a uniformly accelerating observer in flat space-time will feel herself to be immersed in a thermal bath of particles (the so-called 'Rindler' quanta).[1032] Likewise, an observer outside a black hole will detect particles when she is at rest with respect to the hole, but will observe nothing when in free fall. So it appears to be ambiguous whether or not the vacuum state is the state with no particles—it depends on one's state of motion. Nevertheless, as Huggett has indicated, this argument is not conclusive:

> In order to feel the warmth of the vacuum, the accelerating observer must interact with the field, and then she must do work to accelerate against it, exciting the field from the inertial vacuum state. Thus no Rindler quanta can be detected without producing ordinary quanta, and in this case there is no ambiguity: the field is no longer in the vacuum state.[1033]

However, there are further aspects of the treatment of the vacuum state within QFT which have motivated the claim that the content of the theory cannot be exhausted by a particle ontology. Thus, according to Cao, "[i]f we take particles as the primary entities, then the vacuum can only mean a state of nothingness", whereas, "… if we take the fields as the primary entities, then the vacuum, as a ground state of a quantum field, designates a state of a substantial substratum".[1034] The so-called 'virtual particles' that arise within QFT can then be accommodated within this metaphysics, since from this perspective the vacuum can be conceived of as a "… cauldron of interacting particles, each having only a fleeting, or *virtual* existence".[1035] The idea here seems to be that conceived as a substantial substratum, the vacuum

[1030] Redhead 1983.

[1031] Redhead 1988.

[1032] See Cao op. cit., p. 9.

[1033] Huggett op. cit., p. 634.

[1034] op. cit., pp. 9–10.

[1035] DeWitt 1999, p. 178.

state can accommodate the existence of virtual particles as—ontologically—fluctuations of that state/substratum.[1036]

Now, of course, the defender of a particle ontology could insist that this ontology is sufficiently elastic as to accommodate virtual particles as well as 'real' ones. In particular, if one considers how interactions—the scattering of one electron off another, say —are treated in terms of the '*S*-matrix' formalism, in which the scattering is represented by a single operator, the *S*-matrix, which can be expanded as a kind of power series in terms of the interaction Hamiltonian, then creation and annihilation operators appear which are 'naturally' understood to yield the creation and annihilation of particles which do not feature in either the initial or final states.[1037] Diagrammatically the situation can be represented by the famous Feynman diagrams which represent the terms of the *S*-matrix expansion by means of lines and vertices. In particular, every interaction is represented by a vertex and between such vertices 'internal' lines may be drawn representing the virtual particles (which are effectively created at one vertex and annihilated at the other). Historically, the *S*-matrix has been understood as a rival to the QFT programme and the Feynman diagrams in particular associated with a particle ontology.[1038]

However, there are problems here too. As Teller has pointed out, one should not forget that the expression represented by a Feynman diagram is only a component of a much larger perturbative expansion.[1039] And although it may appear tempting to interpret the events of creation, propagation and annihilation as 'parts' of a larger process, these are not 'mereological parts', in the sense that the legs of a chair are such parts of the chair as a whole, but mere 'analytic parts' relative to a basis. The analogy he draws is with Fourier decomposition: just as one would be reluctant to accept the sine and cosine wave forms into which a wave can be decomposed by Fourier analysis as *parts* of that wave, so one should not interpret realistically the propagators belonging

[1036] Cao also claims that the Casimir effect—which shows that the energy density of the vacuum is negative, rather than zero—is a further indication of the substantial nature of the vacuum (ibid., p. 10).

[1037] See the outline in Teller 1995.

[1038] However, Kaiser argues that this association is in fact problematic and that the diagrams do not uniquely endorse the particle picture (Kaiser 1999).

[1039] Teller op. cit. Teller refers to the diagrams as components of a 'superposition' but it has been argued that this is misleading, since this case is unlike that of the expansion of a state in terms of energy eigenstates, where the components of such an expansion undergo evolution. In a perturbative expansion none of the terms represent possible evolutions—the zeroth term, for example, is the solution to the free Hamiltonian and hence cannot represent a solution to the Hamiltonian. This further emphasizes the point that these terms are merely artefacts of perturbation theory and thus their interpretation in terms of virtual particles is untenable. We are again grateful to one of the referees for pressing this point.

to the internal lines in Feynman diagrams since each is but one component in an expansion which extends over all the different space-time points where the relevant vertices could be, then over all distinct diagrams of the same order, then over all diagrams of all different orders that contribute to a process, and finally, within the *S*-matrix itself, integrated again over all of space-time!

Weingard is even more explicit in asserting that virtual particles do not exist.[1040] His reason is that it is not just that what we have is a huge superposition, but that neither the number nor even *kinds* of virtual particles are 'sharp'. Consider first quantized quantum mechanics, under the standard interpretation: if the state of the system is described by a superposition of eigenvectors of some observable, then that observable cannot be said to have a 'sharp' value. It is only if the system can be considered part of an ensemble describable (under the ignorance interpretation) as a mixture, that it is possible for the observable to be 'sharp'. But the latter is not possible within the *S*-matrix framework, since the relevant scattering amplitude is a sum of terms, which means the scattering probability is the square of a sum—characteristic of a superposition—rather than the sum of squares characteristic of a mixture.

Cao, on the other hand, regards the extension of the particle ontology to accommodate 'virtual' particles as fundamentally untenable, not for these reasons but because, he insists, "… the very idea of fluctuations can only be applied to some state of some entities, or some substance, or some substratum, but not to the state of nothingness. The properties of something may be fluctuating, but 'nothing' cannot be fluctuating".[1041] However, this appears to beg the question against the particle ontology: the central point is to conceive of the fluctuations *not* as fluctuations of a substantial substratum but in terms of short-lived and highly constrained virtual particles. From the substantivalist perspective of course 'nothing' cannot fluctuate, but the whole point of this extension of the particle framework is to conceive of the vacuum as a seething 'cauldron' of virtual particles which effectively embody the fluctuations.

It is primarily for the reasons set out by Teller and Weingard that Redhead abandoned his earlier claim that within QFT there exists a kind of underdetermination between field and particle ontologies and he now concedes that his 'extended' particle interpretation is not really a particle interpretation at all.[1042] Nevertheless, granted the above arguments, we do not have an alternative interpretation, nor an explanation of the tendency to interpret the Feynman diagrams and corresponding formalism realistically. As Redhead puts it, there

[1040] Weingard 1982 and 1988.

[1041] op. cit., p. 10.

[1042] Redhead 1988, p. 17.

still remains the 'particle grin'—the fact that what we observe in measurements appears to mesh with some form of particle ontology and it is to this that the defender of such an ontology can always appeal, just as she did in the case of individuals in standard quantum mechanics.

On precisely this point Harré has argued that the 'drift' towards a particle ontology that the Feynman representations favour is not merely the result of convention or some psychological imperative to regard such representation as iconic, but reflects the profound dependence of physical science on experimental practice.[1043] The idea is that the exigencies of such practice effectively 'shape' the concepts of our theories. So, for example, the 'parsing' of certain terms in the relevant mathematical expressions as the propagators of virtual particles is tied to the similarities between the lines of Feynman diagrams and bubble chamber tracks. This 'parsing', however, should not be understood in a straightforward realist sense, as the above considerations demonstrate. Instead we should begin with the 'basic insight' that whatever the ontology of QFT is, some aspect of its causal structure must be manifested in the behaviour of the relevant apparatus. Putting this insight in the terms of our account, what we observe in the scintillation screens and bubble chambers are flashes and tracks which lead us to apply a particle ontology with the underlying metaphysics of Space-Time Individuality.[1044] We then attempt to import that ontology and metaphysics into the quantum realm and further, to extend it to the 'virtual' particles via Harré's 'parsing' of the relevant amplitudes, and, of course, we come a cropper over the sorts of considerations—both physical and philosophical—already noted. It is for this reason that the individuality we ascribe to the entities concerned is only a 'mock' form as Toraldo di Francia puts it. Nevertheless, given its rootedness in experimental practice, the 'particle grin' cannot be dismissed.

However, there is a further problem, which reveals not only the dangers of 'importing' Space-Time Individuality but the inappropriateness of applying it in the first place; this is the famous non-localizability of particles within

[1043] Harré 1988.

[1044] Harre himself chooses the metaphysics of 'affordances', where these are dispositions of things suitably relativized to that with which they interact (ibid., p. 67). Thus whatever the ontology of QFT is, it 'affords' particle-like behaviour, relative to scintillation screens and bubble chambers. Particle talk is thus affordance talk, with the distinction between 'real' and 'virtual' particles understood in counterfactual terms: "A real particle is the referent of an expression for which the … affordance in an experiment motivated by that expression and others which go with it, is a track. A virtual particle is the referent of an expression, analogous to that for a real particle, which, could we construct a track-yielding apparatus, would afford a track or tracks" (ibid., pp. 70–1).

QFT, mentioned above. This aspect features prominently in recent debates, principally due to the work of Malament who argued that, given the standard understanding of relativity theory, no relativistic quantum theory can accommodate the detection of a particle localized in a finite region, a result which appears to remove even the 'grin' of the particle framework.[1045] In effect, what one can show is that any localized particle state will have peculiar and counterintuitive properties, such as superluminal propagation. One possible response, then, is to reject Malament's conditions. Butterfield and Fleming, for example, argue that superluminal propagation does not lead to causal contradictions, nor does it conflict with available empirical data and thus should not be ruled out.[1046] Hence, we can have localizability but at a price, namely the strange properties possessed by the relevant states. In particular, it is a consequence of Butterfield and Fleming's view that the property 'being localized in a certain region of space' need not be detectable within that region. But then, as Halvorson and Clifton remark, in their defence of Malament's claim, this appears to undercut the principal utility of the notion of localizability, "… which is to indicate those physical quantities that are operationally accessible in a given region of spacetime".[1047]

Indeed, they query what the motivation for assigning observables to spatial regions could be, if not to indicate what is locally measurable! But what about the experimental basis for the 'particle grin'? What about our experiences of 'localized' events? Halvorson and Clifton's answer is abrupt: "These experiences are illusory!".[1048] They argue that entities can be 'well enough' localized within QFT so as to give the appearance of being localized. In other words, entities are localized "for all practical purposes".

Again, we can cast this debate in our terms: Butterfield and Fleming's manoeuvre of allowing for non-detectable localizability could be seen as analogous to the introduction of some form of TI for indistinguishable quantum particles. So in this context, Halvorson and Clifton's query above conflates a possible motivation to do with distinguishability with that to do with individuality: if one takes the primary metaphysical function, as it were, of localizability as that of yielding individuality, rather than distinguishability, then one might be prepared to stomach the strange properties of localizable states. Of course, the metaphysical advantage of STI (Space-Time Individuality) over the likes of haecceity and primitive thisness was supposed to be that it could

[1045] Malament 1996.

[1046] Butterfield and Fleming 1999.

[1047] Halvorson and Clifton 2002, p. 7.

[1048] Ibid., p. 20.

underpin both individuality and distinguishability. What the Butterfield and Fleming position does is to remove that advantage and put STI on the same metaphysical level, as it were, as the alternatives.

Finally, it has been suggested that when we attempt to take into account the presence of interactions, the whole framework of QFT on which the above is based falls down and the 'particle picture' is fatally undermined. The crucial result here is Haag's theorem[1049] but how this result is to be understood and how one might respond to it are issues that are open to debate.[1050] Without going into the technical details, it appears to demonstrate that it is impossible to consistently represent interacting QFT within the standard mathematical formalism. Insofar as this is the formalism assumed in the above discussions regarding the 'particle picture', the implication is that the latter must be abandoned. Haag's proof itself can be most clearly presented as a *reductio ad absurdum*:[1051] we begin by supposing that free fields and 'actual' fields incorporating interactions are unitarily equivalent and satisfy the canonical commutation relations. It can then be shown that the vacuum states of the two representations of these relations would in fact have to be the same. It follows from this that the free fields must satisfy a different set of commutation relations than the 'actual' ones, contrary to the initial supposition. The conclusion then is that 'actual' and free fields belong to unitarily inequivalent representations of the canonical commutation relations and thus the 'actual', *interacting*, fields cannot be defined using the same canonical commutation relations as the free fields; or, in other words, the interaction Hamiltonian cannot be defined using the same Hilbert space as the free fields.

Opinions on this theorem vary, from taking it to be a mere inconvenience, or an artefact of the mathematics, or trivial, to regarding it as deep and as a serious foundational problem. Teller, at least, is quite up front about it: "… my exposition will proceed along lines almost universally accepted by practitioners of the theory, disregarding Haag's theorem".[1052] As attractive as such a stance might be, the issue of interactions cannot be avoided completely. Bain has recently indicated a possible way forward which ties in nicely with our considerations regarding the particle 'grin'. He argues that one can side-step the implications of Haag's theorem by adopting the 'LSZ' asymptotic approach from which a notion of particle appropriate to the context of interactions can be extracted.[1053] Again we shall not explore the technical details here, but the

[1049] A standard reference work is Arageorgis 1995.

[1050] In what follows we shall rely heavily on Lupher forthcoming.

[1051] See Lupher op. cit. p. 8.

[1052] Teller 1995, p. 116.

[1053] See Bain 2000.

central idea is to examine how the theory is formulated in the interaction context and use such a formulation as the basis for the particle interpretation, rather than try to extend a conception derived from free field considerations into this context (which is precisely what Haag's theorem rules out). What this gives us is a more general conception of 'particle' as a system which is free for all practical purposes at asymptotic times. In particular, on such a conception, the issue of whether there is a corresponding number occupation operator is understood to be irrelevant.[1054]

Of course, as Bain himself notes, it might be objected that a 'for all practical purposes' basis is not metaphysically robust enough to ground an appropriate notion of particle. In particular, within this asymptotic approach, particles possess finite exponential 'tails' which may span space-like separated regions and hence throw up concerns about localizability again. Bain's response is simply to point out that such 'tails' are the inevitable result of the persistence of interactions and to insist on a particle conception in which they are not present is to beg the question against his approach and allow Haag's theorem to hobble the particle conception from the outset. We can extend this response via our remarks above: if we underpin the particle picture with some form of TI and abandon STI (Space-Time Individuality) then perhaps we can accommodate the failure of the intuition that localizability is something a particle must possess, whatever state it is in. Furthermore, the objection that whereas 'for all practical purposes' localizability is good enough for the practising physicist, it is too flimsy a basis for the philosopher of physics, precisely begs the question against the view espoused by Toraldo di Francia. In effect this view runs counter to that of Halvorson and Clifton's above and Bain's 'asymptotic' conception can be taken as accommodating the particle 'grin'. From this perspective, 'for all practical purposes' is simply the best we can do.

The upshot of these considerations, then, is that at least certain aspects of the particle ontology can be maintained, but at a (metaphysical) price. And, at the very least, *any* putative metaphysics for QFT is going to have to accommodate the 'particle grin' we appear to observe. Having outlined some of the technical issues involved in the physics, let us now consider the metaphysical alternatives in a little more detail.

[1054] 'Asymptotic' particles possess such operators, whereas 'interacting' particles do not. It is important to note Bain's insistence that he is not seeking to establish the particle interpretation to the exclusion of the field picture—both feature in the ontology of QFT.

## 9.2 METAPHYSICAL OPTIONS

Can all these different aspects of a quantum field be accommodated within a single metaphysical framework? Obviously one's preference for the form of the latter will depend on which of the above aspects one gives particular prominence. Let us begin with the view—advocated by Cao, for example—that the above problems decisively rule out the particle picture and that QFT should ultimately be regarded as a *field* theory.

We have touched on the metaphysics of fields above and we recall the two standard alternatives: fields as substances and fields as properties. With regard to the former, Simons, in his taxonomy of possible metaphysical frameworks, dismisses substance on the grounds that we have lost "… the notion of identifiable and reidentifiable individuals in quantum theory".[1055] However, as we have insisted, one can maintain the view of quantum particles as individuals and, furthermore, as individuated via some substantival notion, albeit at a price. More importantly, perhaps, the Received View alluded to in Simon's analysis applies to particles and there is nothing in this to prevent *fields* being regarded as substantial.

The alternative is to regard field quantities as properties of the underlying space-time. The cost in this case, as we noted in Chapter 2, is that one must then adopt a substantival view of space-time, on pain of circularity (since on a relationist view, space-time would be nothing more than a set of relations holding between material objects which are ultimately nothing more than quantum field theoretic properties of space-time …). Thus, it would seem that we cannot get away from substance in one form or another.

There are various possible ways of evading this dilemma. Consider again the view of fields as sets of properties and let us ask why space-time needs to be invoked at all on such a view. The answer, it would seem, is obvious: properties are universals and require instantiation if they are to serve as part of our physical ontology. Hence, if we reject physical substance, our field properties must be instantiated at the points of space-time. Perhaps, then, we could avoid this consequence if our metaphysical conception of a field employed something akin to properties but without the latter's 'universal' nature. The notion of a 'trope' might fit the bill.

[1055] Simons 2002, p. 36.

A trope can be thought of as a non-repeatable property instance. Thus, instead of regarding the mass of a particular body at a particular time as the instantiation of the universal property mass, we understand it as a kind of individual property: *that* mass. Likewise, instead of considering field quantities as property instantiations at particular space-time points, we might take each such quantity as a trope. The advantage—or so it is claimed—is that, with both particulars and properties constructed out of, or reduced to, tropes, we get a parsimonious one-category ontology. There are costs, however. First of all, tropes are typically taken to be dependent entities, as the example of mass suggests: there can be no mass, understood as a trope, without something which possesses this mass. For Simons this is enough to rule them out as the fundamental constituents in a field-theoretic context:

> If something is dependent, then surely, as Bolzano argued, there must be at least one thing that is independent, even if it is the whole world. So there cannot be a tropes-only ontology: the world is not a trope, and the world is not nothing.[1056]

In the case of field quantities, it would seem, they must still be dependent on space-time points, and our circularity re-appears.

However, this is too quick. Tropes can form bundles or clusters and a member of such a cluster may turn out to be dependent on other tropes, rather than something non-tropic. Indeed, this dependence can be mutual and in such cases a group of tropes must either all exist or none do. Furthermore, tropes may require other tropes as members of a kind and in such cases, instead of 'founding' we have 'generic dependence', with the tropes generically required forming a 'halo'. Perhaps, then, an entire quantum field, as a kind of 'global' entity, might be understood as such a cluster of generically dependent tropes and there would then be no need to posit something non-tropic in order for the tropes to be dependent upon.

Unfortunately, there is a second cost associated with such a move which has to do with the motivation for adopting any such metaphysical framework in the first place. Surely, one reason we introduce such frameworks is to effect a certain ontological simplicity. In the case of tropes, we gain such simplicity along one dimension, as it were, by having only one category but we lose it entirely along the dimension of quantity of distinguishable members of our categories; that is, instead of having a limited number of universal properties instantiated at (indistinguishable) space-time points, each and every

[1056] Simons 2002, p. 44.

such instantiation is now effectively reified. We now have as many (non-denumerably many!) individual tropes as there are space-time points and one might well wonder what has been gained—metaphysically speaking—from such an ontology.

Auyang has suggested an interesting way of breaking out of the above circularity by insisting that neither the field nor space-time should be given ontological priority; rather they both emerge together out of the 'world structure'.[1057] Thus she proposes a view of space-time according to which it is absolute, in the sense that it is presupposed by the 'concept' of individual things, but not substantival, since it is ultimately structural.[1058] It is this space-time structure that individuates the 'field events' of a quantum field and such events satisfy the condition of "having their own identity".[1059] And these events, of course, compose the quantum field.[1060] Field events are thus individuated structurally within 'the whole' and Auyang is clear that neither the space-time structure nor the event structure should be given ontological priority: "[t]he event structure and the spatio-temporal structure of the objective world emerge together".[1061]

Auyang's ontology has been criticized by Bartels for its lack of clarity in specifying what is meant by a 'field event'. Auyang herself identifies such events with the local field operators at a given space-time point.[1062] However, this is problematic, as these operators do not specify 'concrete' local events; rather they,

> … describe the dynamics of quantum systems in a very general sense, as a type of general dynamical condition that applies to a large set of quantum systems.[1063]

In other words what these operators specify are not specific local events but rather *kinds* of events. In order to describe 'concrete' events, one needs to apply the field operator to a specific state vector. One could insist that by 'concrete' events we mean the results of specific measurements of some physical value.

[1057] Auyang 1995. There are historical antecedents for such a view in Eddington's work; see French 2003. More recently, as we noted in Chapter 2, Stein proposed that space-time is an aspect of the 'structure of the world'.

[1058] Ibid., p. 138. This is strongly reminiscent of Newton's view—again mentioned in Chapter 2—of space as an affection of a thing *qua* thing', understood by Stein in terms of it being an aspect of the nature of thinghood.

[1059] Ibid., p. 122.

[1060] Ibid., p. 129. A little more formally, they are identified by a parameter, *x*, of the relevant base space in a fibre bundle formulation, and divided into kinds via the appropriate group (ibid., pp. 130–2).

[1061] Ibid., p. 135. An indication as to how this view might be accommodated within a form of structural realism can be found in French and Ladyman 2003.

[1062] Auyang 1995, p. 129.

[1063] Bartels 1999, p. 179.

But, of course, this cannot be what we mean by a field event since such definite values cannot be ascribed to quantum systems as intrinsic properties.

Finally, one might be tempted by a compromise interpretation under which field events are taken to refer to *operations* of some local field operator on a state of the field system at some space-time point. As an example of such an operation, consider the excitation of a field mode of a certain type. However, Bartels argues, this will not work for reasons to do with the prohibition on localizable particle states in QFT, noted above.[1064] Bartels understands this to imply that there can be no strictly local operations on particle states. Hence, he concludes, if field events are understood in terms of such local operations, they cannot be the 'distinguished fundamental entities' of QFT.

The problem appears to be the reliance on some form of STI in order to render the field events 'concrete'. Bartels' own solution is to turn to Davidson's theory of events according to which an event is to be identified, not via its spatio-temporal location, but rather in terms of its 'causal location' within the causal net of the world.[1065] This theory can then be applied to the physical systems of QFT:

> To think of the systems as 'events' takes account of the fact that the systems are not enduring substances with an intrinsic individuality (primitive thisness), but simply instantiations of quantum states. Instantiations of properties, pointlike or spatiotemporally extended, are what I call '*events*'.[1066]

In order to motivate this ontology, Bartels offers the following example: consider the decay of some particle in a high-energy physics experiment. The original state of the particle is identified via the identification of the decay products. Hence, Bartels claims, the future causal history of the particle is used to individuate the state. Spatio-temporal localization cannot do this because of the non-localizability problem. Furthermore, he continues, even if the spatial separability of the states was such that we could distinguish them at a given time, we would lose this distinguishability at later times as the state spreads out. The loss of identity inherent in this process is emphasized in the following passage:

> If we count the state at t as the same event as the state at *t'* [where *t'* is later than t], then its identity would be lost between t and *t'*; if we count them as different events, however, then the state at *t'* will have no identity.[1067]

[1064] Bartels 1999, p. 180.

[1065] Ibid., p. 181.

[1066] Ibid., pp. 181–2; note the explicit concern with avoiding attributions of individuality here.

[1067] Ibid., p. 182.

Hence quantum systems are better identified by their causal histories than by spatio-temporal localization.

This attempt to substitute an 'event' ontology for objects is interesting but not unproblematic. First of all, it may well lead us back to familiar onto- logical categories, depending on the metaphysical analysis adopted. Thus, for example, it seems plausible to maintain that events 'have'—in some sense—participants and indeed it has been argued that events ontologically supervene upon their participants. Obviously, if the latter are taken to include objects, in some metaphysical form or other, then Bartels' move will not enable us to escape the sorts of issues previously discussed. In particular, as is well known, Strawson argued that successful re-identification rules out a pure events-based ontology since such re-identification requires a stable frame of reference, as provided by objects.[1068] Indeed, there are ontologically 'conservative' accounts of events which take them to be nothing more than exemplifications of properties of objects at particular times. Of course, these accounts have themselves been contested and even inverted, with objects understood as ontologically dependent on the events in which they 'participate'. Alternatively, one might adopt a 'third way' kind of position in which events and objects are granted equal ontological status. It is not our intention to delve any further into these debates here; we simply wish to emphasize that there is a lot more work to be done before an events-based foundation for QFT can be accepted.

Secondly, questions arise about how to represent the relationship between the quantum state and a specific representation in terms of quanta, within the events framework. Bartels' suggestion is that this can be modelled by the relationship between events and 'states of affairs', since a particular event can be represented by different states of affairs depending on how the event is characterized. Likewise, a particular occupation number state can be thought of as expressing a state of affairs which represents the event that the system is in a particular quantum state, relative to a chosen basis.[1069] Now, however, we must consider the further issue of the ontological nature of the states of affairs themselves. Armstrong takes them to be special complex entities which are composed of an attribute and the appropriate number of terms. Thus Bartels considers the event "the pouring of a dose of arsenic into a glass of water" as represented by the state of affairs "John kills Agatha by poisoning".[1070] The problem is that in specifying the state of affairs in this example we have had to invoke the particulars John and Agatha, plus the

[1068] Strawson 1959.

[1069] Ibid., p. 183.

[1070] Ibid.

relation '*x* kills *y* by poisoning', so why not just stick to the relevant particulars and their relevant attributes to begin with? This is precisely the reason why Simons dismisses ontologies based on states of affairs and the like as inadequate in this context.[1071] In particular, if we are considering a particular occupation number state, then we will cite the relevant number of quanta but if we are going to do that, why bother invoking the notion of a 'state of affairs' in the first place?

Bartels might well respond that the problem with quanta, of course, is their lack of identifiability but, as we have argued in the previous two chapters, that should not be seen as a problem at all, either metaphysically or formally. Of course, when it comes to the 'concrete' events that we observe, these can be identified and if one were to accept the argument against spatio-temporal localizability within QFT, then Bartels' suggestion of appealing to causal location offers an appropriate way of establishing this identification. However, as he himself puts it, the quantum systems themselves are hidden from this measurement context as if by a 'veil' and one must be wary of transferring this 'pseudo-individuality'—whether grounded in spatio-temporal or causal location—from this context into the quantum domain.

Clearly there is further metaphysical work to be done here. Our aim has been simply to suggest that many of the responses to these foundational issues have been motivated by concerns over the individuality of quantum objects and that such concerns can be met without giving up objects *per se*. We shall now consider the possibility that quasi-set theory offers a formal representation of the non-individual quanta of QFT. It is important to note that we shall not give the full details of the formalization here, as this would take us too far into the logico-mathematical specifics. What we shall do is draw on the technical material delineated in the previous chapters in order to indicate how such a formalization might be elaborated.

## 9.3 MODELS AND THE FOCK SPACE FORMALISM

Let us begin by recalling the basic metaphysical features of 'quanta'. Teller makes an explicit contrast with individual particles:

> … things with primitive thisness can be *counted*; that is, we can think of the particles as being counted out, the first one, the second one, the third, and so on, with there being a difference in principle in the order in which they are counted, a difference that

[1071] Strawson 1959, pp. 5–6.

> does not depend on which particle has which properties. By way of contrast quanta can only be *aggregated*; that is, we can only heap them up in different quantities with a total measure of one, or two, or three, and so on, but in these aggregations there is no difference in principle about which one has which properties. The difference between countability and susceptibility to being merely aggregated is like the difference between pennies in a piggy bank and money in a modern bank account.[1072]

We also recall that Hesse explicitly invokes this money in the bank analogy to illustrate what she means by quantum particles lacking self-identity: "[w]e are unable to identify individual electrons, hence it is meaningless to speak of the self-identity of electrons …".[1073] Carrying this over into the domain of QFT, as Redhead and Teller do, suggests a way of responding to the Manin Problem of Chapter 6, namely by regarding aggregates of quanta in terms of quasi-sets.

By doing so we can represent and bring out, philosophically, the distinction emphasized by Teller between 'countability' and 'susceptibility to being aggregated'.[1074] The connection between individuality and countability is often made in the literature. Thus Lowe, for example, in his discussion of primitive substances, situates individuality within the framework of differentiation or distinctness and stresses that the latter must be achieved in such a manner that the individuals can be regarded as countable. A necessary condition of countability is that

> … the items to be counted should possess determinate identity conditions, since each should be counted just once and this presupposes that it be determinately distinct from any other item that is to be included in the count.[1075]

Hence, if a plurality is countable, the entities of which it is composed must possess self-identity, according to Lowe. However, the notion of 'countability' that is invoked here is not at all clear. In particular we need to distinguish *cardinality* from *ordinality* and this is precisely achieved in quasi-set theory (and also in quaset theory as we have indicated earlier). With regard to aggregates of quanta one can have good theoretical and experimental reasons for ascribing a determinate number of quanta to such a collection but without being able

[1072] Teller op. cit., p. 12.

[1073] Hesse 1970, p. 50.

[1074] Barcan Marcus notes that in the foundations of set theory "… there is an ambiguity in the notion of 'gathering together' and 'conceiving a totality' that is reflected in the distinction between an itemization of elements and a statement of conditions for membership" (Barcan Marcus 1993, p. 93). It seems to us that this broadly corresponds to the distinction between 'countability' and 'susceptibility to being aggregated' and correspondingly to that between ordinality and (quasi-) cardinality captured by quasi- (and qua) set theory. As Barcan Marcus goes on to note, "[t]he obliteration of the difference by the logic of set theory does not generate too many perplexities for mathematics …" (ibid.) but it is a different matter when it comes to physical objects.

[1075] Lowe 1994.

to count them, in the sense of putting them in a series and establishing an ordering. As we have seen, we may consider a quasi-set of quantum particles (mainly within the theory $\mathfrak{Q}^m$ described in Chapter 7, which assumes the existence of *m*-atoms) which possesses a cardinal but not an ordinal in exactly this sense—the elements of a quasi-set may be countable, *in the sense of possessing (quasi-) cardinality but not in the sense of possessing ordinality.* It is in this manner, we believe, that Teller's distinction can be formally represented.

What we are attempting to do here is to suggest a formal framework for understanding the notion of quanta and it is our firm conviction that such a framework is absolutely necessary if something more than mere lip service is to be paid to this notion and its underlying metaphysics.[1076] We wholeheartedly agree with Auyang, who says,

> A proper interpretation of quantum theories calls for a general concept of the objective world that adequately accommodates both classical and quantum objects.[1077]

where we would extend the notion of a 'proper interpretation' into metaphysics.

However, focusing on the notion of 'non-individuality' and a 'set' theory suitable for capturing it in this way highlights a two-fold tension with regard to the Fock space approach to QFT, within which the notion of quanta are articulated, and the model-theoretic approach to the latter, advocated by Teller for example.[1078] We can reveal this tension by recalling our discussion of what it is to 'axiomatize' a scientific theory. As we have stated in Chapter 6, the axiomatic basis of a theory basically encompasses three 'levels': (1) the axioms of the underlying logic of the theory, say first-order logic with identity; (2) the axioms of the mathematical apparatus, say Zermelo-Fraenkel set theory; and (3) the specific axioms of the theory itself. [1079]

According to the model-theoretic or 'semantic' representation of theories these axioms delineate the class of models, or more generally, mathematical structures specified by the theory. As Teller puts it, these models are viewed as "… abstract objects embodying just those properties and relations which are

[1076] There is obviously a crucial issue as to what 'understanding' consists in here. At the very least we are sympathetic to Carnap's feeling that "[w]hen I considered a concept or a proposition occurring in a scientific or philosophical discussion, I thought that I understood it clearly only if I felt that I could express it, if I wanted to, in a symbolic language" (Carnap 1963, p. 11).

[1077] op. cit., p. 6.

[1078] Teller 1995.

[1079] Of course there are other possibilities, generated by using a suitable higher-order logic instead of steps (1) and (2), for example, but we are just summarizing the issue here, and there is no loss of generality in accepting our schema.

of interest to us in the physical systems to be modeled".[1080] This is an approach to which we are very sympathetic; however, for these 'abstract objects' to be models, in the strict sense, they must be 'satisfiers' of some sort. According to Suppes, as we indicated in Chapter 7, what they satisfy is a set-theoretic predicate and to axiomatize a theory is to define just such a predicate.[1081] Thus, if axiomatization delineates a class of models, then of course it also involves axiomatization at levels (1) and (2), at least implicitly, since these axioms 'effect' a theory's theorems. In this regard it is perhaps worth recalling Church's remark (made with respect to mathematics) that,

> … for the precise syntactical definition of the particular branch of mathematics it is necessary to state not only the specific mathematical postulates [what we would call axioms of level 3] but also a formulation of the underlying logic, since the class of theorems belonging to the branch of mathematics in question is determined by both the postulates and the underlying logic and could be changed by a change of either.[1082]

This brings us to the Fock space approach, within which (and only within which) quanta can be defined and advocated by Redhead and Teller as being the framework in which non-individuality can be understood in terms of a 'label-free formalism'. However, if this is to be characterized in model-theoretic terms, as Teller intends, axiomatization via a set-theoretic predicate must be assumed, again at least implicitly and, in this particular case, such axiomatization cannot be left as 'an exercise for the reader', as it were, precisely because of the metaphysical issues driving the move to this approach. Let us be specific: standard presentations of the Fock space formalism, such as given by Teller, Geroch, Landau and Lifschitz,[1083] for example, are elaborated within the framework of 'naïve' (Cantorian) set theory which is precisely based upon ontological presuppositions which the metaphysics of quanta supposedly denies! In other words, at levels (1) and (2), classical logic and mathematics have been left intact as the necessary formal underpinning; only the axioms at level (3) have been changed from those of the first quantized approach to Fock spaces. The problem, then, is to consider the adequacy of the class of models these axioms describe and the intuitive motivation which was used for the Fock space formalism, namely the rejection of 'primitive thisness' and individual elementary particles. Furthermore, it is not only a question of finding a more adequate language for expressing the axioms of the theory. If quanta can be (strongly—that is, non-classically) indistinguishable entities of a sort, then

[1080] op. cit., p. 4.

[1081] Suppes 1959, 1967, 2002.

[1082] Church 1956, p. 317.

[1083] Teller 1995, Geroch 1985, Landau and Lifschitz 1959.

the models should not be constructed in standard set-theoretical terms, but rather within a mathematical framework which enables us to consider such indistinguishability.

## 9.3.1 A Suppes Predicate for QFT

Let us sketch here a minimal nucleus of what we call a *Fock Structure*. In order to do justice to what we have said above with respect to the use of a 'more adequate language' for speaking about indistinguishable quanta, we shall work within quasi-set theory as presented in Chapter 7. But since all the concepts to be considered are 'classical' (Hilbert spaces etc. ), we will be working mainly with the 'classical part' of quasi-set theory. Hence for simplicity we will use the standard set-theoretical notation instead of the one peculiar to the theory $\mathfrak{Q}$, since the translation can be made without difficulty. We shall explicitly indicate when quasi-sets must be considered.

Before we begin, we recognize that the reader may justifiably be asking herself why we are emphasizing the importance of quasi-sets if we plan to use 'classical' concepts. The reason is simply that we will follow the standard way of approaching the subject by means of Fock spaces and shall reserve quasi-sets for giving a tentative interpretation of the relevant vectors. Of course it would be interesting to show how QFT can be fully constructed within quasi-set theory. An indication was provided in Chapter 7 for standard quantum mechanics, where we have shown that in using such a mathematical language, certain strong assumptions, like the Indistinguishability Postulate, do not need to be assumed. However, the full details involved in developing QFT within $\mathfrak{Q}$ would take up several more chapters, so we shall only sketch the framework and its philosophical underpinnings here.

We shall begin with some definitions. Let us designate by a Fock Structure the ordered triple

$$\mathfrak{F} =_{\mathrm{df}} \left\langle F, a_i^{\dagger}, a_j \right\rangle_{i,j \in I}, \tag{9.3.1}$$

where $F$ is a Fock space (understood as a direct sum of tensor products of Hilbert spaces in the usual sense),[1084] while $a_i^{\dagger}$ and $a_j$ are linear operators on $F$ that satisfy the following conditions:

[1084] For details, see Geroch, op. cit., Chapter 21; Teller op. cit., Chapter 3; but see below. Alternatively, we could consider in the above structure only one kind of operator $a_i^{\dagger}$ or $a_j$ and define the other class in terms of the chosen one. However, we shall simplify the exposition.

- for every $i \in I$, $i \in I, a_i^\dagger$ and $a_i$ are Hermitean conjugates of each other.
- these operators obey the standard commutator and anti-commutator relations.[1085]

By means of such a definition, we can construct an appropriate 'Suppes predicate' for the mathematical counterpart of quantum field theory, since all models of non-relativistic QFT can be obtained in this way.[1086] So, we may say that $X$ is a *Fock Space Quantum Field Theory* if $X$ satisfies the following set-theoretical predicate ((1) and (2) refer to the two items of the above definition):[1087]

$$\mathrm{Fock}(X) =_{\mathrm{df}} (\exists F)(\exists a_1^\dagger, a_2^\dagger, \ldots)(\exists a_1, a_2, \ldots) \times (X = \left\langle F, a_i^\dagger, a_j \right\rangle \wedge (1) \wedge (2)).$$

(9.3.2)

It would be necessary to verify that such a formula is transportable in the sense of Bourbaki, which we assume here without proof. Intuitively, this means that its constituent parts can be substituted by suitable replacements by means of isomorphisms.[1088] Of course this predicate gives an idea of only the mathematical counterpart of QFT, for we have not made reference to the domains of application. But we recall that standard formulations of QFT do not make direct reference to them either, for when we say that $\psi$ refers to a collection of quanta, no adequate semantics is formally constructed; we shall return to this point below.

Before we do, let us consider briefly the nature of the models of the predicate defined above. So, let us take $\mathscr{H}$ to be a complex Hilbert space with orthonormal basis $\{ |v_i\rangle \}$ ($i$ = 1, 2, 3, …),[1089] where the $v_i$ are eigenvectors of some maximal (non-degenerate) Hermitian operator $A$ defined on $\mathscr{H}$. Then

[1085] See Merzbacher 1970, p. 515.

[1086] Cf. van Fraassen 1991, p. 448. As exemplified for instance in Mattuck 1967, Chapter 7, with the help of 'raising' and 'lowering' operators (the operators $a_i^\dagger$ and $a_i$ above) all of second-quantized quantum mechanics can be constructed.

[1087] Strictly speaking, the definition of 'Suppes predicate' is not exactly the same as we have used above. Such a definition was given by da Costa and Chuaqui in their 1988 paper (as recognized by Suppes himself—see his 2002, p. 33). da Costa and Chuaqui's definition makes use of free variables for principal sets in Bourbaki's original sense (Bourbaki 1968, Chapter 4) while we have used only 'secondary sets' instead for the relevant Fock spaces. But our approach can be seen as a 'relativized' definition of Suppes predicate. We thank Professor da Costa for this remark, made in an informal conversation.

[1088] Bourbaki 1968; an application of Bourbaki's concept of 'species of structures' to physical theories can be found in G. Ludwig's and E. Scheibe's works (Ludwig 1990, Scheibe 2001).

[1089] The space $H_1$ mentioned in an earlier footnote.

we take $F$ to be the direct sum

$$F = {}_{\mathrm{df}}\bigoplus_{n=0}^{\infty}\mathcal{H}^n,$$

(9.3.3)

where $\mathcal{H}^0 =_{\mathrm{df}} \mathbb{C}$ (the set of complex numbers seen as an Hilbert space), $\mathcal{H}^1 =_{\mathrm{df}} \mathcal{H}$, $\mathcal{H}^2 =_{\mathrm{df}} \mathcal{H} \otimes \mathcal{H}$ and so on.[1090] In this Fock space vectors such as $\alpha_1 = v_1 \otimes v_2$ and $\alpha_2 = v_2 \otimes v_1$, $(v_1 \neq v_2)$ appear, since they are both elements of $\mathcal{H}^2$. But this introduces the 'surplus formal structure' rejected by Redhead and Teller.

More appropriate models can of course be constructed, such as the *boson model*: Here we take $\mathcal{H}^n$ as above but define on this space a function $\sigma^n$: $\mathcal{H}^n \rightarrow \mathcal{H}^n$ $(n = 1, 2, \ldots)$ by

$$\sigma^n(v_1 \otimes \cdots \otimes v_n) = (1/n!)\Sigma_P P(v_1 \otimes \cdots \otimes v_n),$$

(9.3.4)

where $P$ is an element of the permutation group of $\mathcal{H}^n$. We now define

$$\mathcal{H}^n_\sigma = {}_{\mathrm{df}}\{\sigma^n(v) : v \in \mathcal{H}^n\}$$

(9.3.5)

and

$$F_\sigma = {}_{\mathrm{df}}\bigoplus_{n=0}^{\infty}\mathcal{H}^n_\sigma,$$

(9.3.6)

where, once more, $\mathcal{H}^0_\sigma = \mathbb{C}$. Clearly $\mathcal{H}^n_\sigma$ is the symmetric subspace of $\mathcal{H}^n$.[1091] The *fermion model* can be obtained in a similar way by defining functions $\tau^n$: $\mathcal{H}^n \rightarrow \mathcal{H}^n$ such that

$$\tau^n(v_1 \otimes \cdots \otimes v_n) = (1/n!)\Sigma_P s^p P(v_1 \otimes \cdots \otimes v_n),$$

(9.3.7)

where $s^p$ is the *signature* of $P$, that is, $s^p$ is $+1$ if $P$ is even and $-1$ if $P$ is odd.[1092] Then, by analogy with the symmetric case, we define the anti-symmetric subspace of $\mathcal{H}^n$ as

$$\mathcal{H}^n_\tau = {}_{\mathrm{df}}\{\tau^n(v) : v \in \mathcal{H}^n\}$$

(9.3.8)

and

$$F_\tau = {}_{\mathrm{df}}\bigoplus_{n=0}^{\infty}\mathcal{H}^n_\tau.$$

(9.3.9)

[1090] Geroch op. cit., Chapter 21.

[1091] van Fraassen op. cit., p. 437.

[1092] Ibid., p. 386.

The problem now is to define the operators $a_i^\dagger$ and $a_j$. Let us define, for each $\alpha \in \mathscr{H}'$, $\alpha = v_{i_1} \otimes \ldots \otimes v_{i_n}$, arithmetic functions $n_\alpha$, taking values on the set $\{1, 2, \ldots, n\}$ and ascribing to each $k \in \{1, 2, \ldots, n\}$ the quantity of sub-indices among the $i_j$ which equal $k$. The numbers $n_\alpha(k)$ are called the *occupation numbers of* $v_{i_j}$ *in* $a$. It can be shown that $n_\alpha(k) = n_\beta(k)$ if and only if $\sigma^n(\alpha) = \sigma^n(\beta)$ $(i = 1, \ldots, n)$. Then we can pay exclusive attention to the occupation numbers and write them as an $n$-tuple $\langle n(1), n(2), \ldots, n(n)\rangle$ (where the sub-indices have been omitted and $\sigma\, n(\mathrm{i}) = n$) for every basis vector of $\mathscr{H}'$ (and similarly for the fermion model). Thus it has been claimed that in this way we arrive at an 'individual-label-free notation', since the vectors are mentioned by their occupation numbers only.[1093]

It is worth noting that the basis vectors of the Fock space vectors may be represented by sequences whose $(n + 1)$th element is precisely $\langle n(1), n(2), \ldots, n(n)\rangle$ and whose other elements are zero vectors. Then, by paying attention exclusively to their occupation numbers as above, we need not make reference to the full sequence but only to this $(n + 1)$th term. Hence the basis vectors of the symmetric subspace can be represented as usual by:

$$|n\rangle = |n(1)n(2)\ldots n(n)\ldots\rangle.$$

(9.3.10)

We insist that this notation *stands* for a certain vector in the Fock space, which is uniquely characterized, as noted by van Fraassen,[1094] jointly by the occupation numbers $n(i)$, whose sum equals $n$ (this $n$ indicates to which $\mathscr{H}'$ the vector belongs) and by the values of the n(i) themselves, which indicate to which vectors in the basic Hilbert space $\mathscr{H}$ the 'true' vector is linked. In this way, we may say that $|\ 2\ 0\ 7\ 0\ 0\ \ldots\rangle$ represents a state (a vector) with two quanta with eigenvalue $v_3$ and seven quanta with eigenvalue $v_7$ with respect to the observable $A$ from which we have picked up the discrete orthonormal basis $\{|v_i\rangle\}$ of $\mathscr{H}$.[1095]

We are now in a position to define, for each $i = 1, 2, \ldots$:

$$a_i^\dagger |n(1)n(2)\ldots n(i)\ldots\rangle = {}_{\mathrm{df}}\sqrt{n(i)+1}\,|n(1)n(2)\ldots(n(i)+1)\ldots\rangle$$
$$a_i |n(1)n(2)\ldots n(i)\ldots\rangle = {}_{\mathrm{df}}\sqrt{n(i)}\,|n(1)n(2)\ldots(n(i)-1)\ldots\rangle.$$

(9.3.10)

[1093] van Fraassen op. cit., p. 438; in the fermion case of course these are either 0 or 1.

[1094] op. cit., pp. 440–1.

[1095] Teller 1995, pp. 37ff.

In the standard case, it can be shown that the $a_i^\dagger$ and $a_i$ have the properties of the 'raising' and 'lowering' operators respectively and, as was remarked above, all the second quantized formalism can be obtained.

A potential problem now arises: the 'label-free notation' mentioned above is achieved by taking the symmetric Fock space (and likewise for the antisymmetric case, of course). But we recall that we began with a Hilbert space $\mathscr{H}$ with basis $\{|v_i\rangle\}$, as van Fraassen, for example, has emphasized and the question is whether this procedure of constructing a Fock space from a labelled Hilbert space avoids the criticism directed by Redhead and Teller to the 'labeled-tensor-product-Hilbert-space-formalism' of the first-quantized approach.[1096]

In order to accommodate quanta, the right way of going about this construction, as both van Fraassen and Teller have suggested, should be to begin with an 'abstract' Fock space in which only the occupation numbers were mentioned. That is, we should construct the Fock space by taking the set of vectors as being the sequences $|n(1)\ n(2)\ \ldots\rangle$. This is precisely what Teller claims by saying that such vectors 'generate' the elements of the direct sum in the definition of the Fock space.[1097] However, from a mathematical perspective this is rather odd, since we cannot define vector space operations on $n$-tuples of natural numbers.[1098]

Perhaps we could understand Teller's claim that $|1\ 1\ 0\ \ldots\rangle$, $|2\ 0\ 0\ \ldots\rangle$, … for example, *generate* a Hilbert space only by regarding these $n$-tuples as the vectors they represent in $\mathscr{H}^0$, but then we arrive again at the Hilbert space $\mathscr{H}$ and the concern arises that we are still not free from primitive thisness.

Although we think that the correct direction has been indicated by both van Fraassen and Teller, their approach still commits us to what we call Weyl's strategy: we take *individuals* of a kind (elements of a set, or vectors in the relevant spaces) and postulate that permutations are not observable. In the present case, we have taken a labelled Hilbert space and then have supposed that the vectors $|n(1)\ n(2)\ \ldots\ n(n)\ \ldots\rangle$ enable us to say that we are working in a label-free notation. This procedure, as we have said, effectively 'masks' the

[1096] Cf. also Huggett op. cit., pp. 73–4.

[1097] Cf. Teller op. cit., p. 39.

[1098] That is, if we try to follow Teller's suggestion that (for instance) the vectors $|1\ 0\ 0\ \ldots\rangle$, $|0\ 1\ 0\ \ldots\rangle$, … should span the Hilbert space $\mathscr{H}^1$, while $|1\ 1\ 0\ 0\ \ldots\rangle$, $|1\ 0\ 1\ 0\ \ldots\rangle$, $|0\ 1\ 1\ 0\ \ldots\rangle$ span the space $\mathscr{H}^2$ and so on (Teller 1995, p. 39), then the vector-space operations must be defined for such vectors. But taking any field $F$ (for instance, the complex number system $C$), then the product of a vector by an scalar (a complex number) is not well defined; for example, $(3 + 2i)\ |1\ 0\ 1\ 0\ \ldots\rangle$ is no longer a 'vector' in $\mathscr{H}^2$. In short, there is no way of defining the mathematical structure of vector spaces in this way.

individuality of the elements of the sets (the chosen basis vectors), but cannot be said to be the correct mathematical treatment of indistinguishable objects. With respect to the models described above, we suggest that something similar is occurring when we take the symmetric and anti-symmetric subspaces by 'forgetting' the initially attached labels.

## 9.4 QUASI-SETS AND THE OBJECTIVITY OF QUANTA

So, taking into account that $|n\rangle = |\, n(1)\ n(2)\ \ldots\rangle$ only represents a certain vector in the Fock space, then, paraphrasing Teller when he asks for a 'rationale' for the labels of the Hilbert space formalism, typical of first quantization,[1099] we could ask: how are the labels of these vectors (in the Fock space) to be understood? Recall once more that any one of these occupation numbers is related to a pure state in the original Hilbert space $\mathscr{H}$.[1100] In our opinion, there still remains the problem of finding models of QFT in which the non-individuality of quanta is ascribed "right at the start", to use Post's phrase.

Given the above and, specifically, the ontological considerations driving the shift to the Fock space approach, the question, then, is how we are to read the $n$-tuple $|\, n(1)\ n(2)\ \ldots\ n(n)\rangle$ . A clue perhaps is given by Auyang, who writes:

> To say the field is in a state $|\, n(1)\ n(2)\ \ldots\ n(n)\rangle$ is not to say that it is composed of $n(k_1)$ quanta in mode $k_1$ and so on but rather $n(k_1)$ quanta show up in an appropriate measurement.[1101]

This may be correct from the purely physical point of view, but the metaphysics remains unclear, both as to what it is to say that the field is in such a state if it is not to say that it is *composed* of quanta and how it is that 'from' such a state, as it were, the quanta emerge on measurement. In trying to clarify the situation and construct an adequate semantics for the quantum formalism, let us recall our emphasis on the metaphysical presuppositions which underlie not only classical physics but standard logic and mathematics as well. It is such presuppositions which explain in part why these theories encompass what we have called in this book the 'traditional' theory of identity.

In the case of macroscopic, observable objects at least, the relevant semantics is apprehended by us quite directly, without the need for a careful mathematical description of the details involving the domains of application (which for

[1099] Cf. Teller 1995, p. 25.

[1100] Cf. van Fraassen, op. cit., p. 437.

[1101] Auyang 1995, p. 159.

the sake of correctness should be provided, and logicians have given us the details, as is well known). So, for instance, we may say that proper names (of people)—however they are understood philosophically—apply 'directly' (ostensively) to people, without any need for a mathematical description of people's behaviour.[1102] When it comes to unobservable phenomena, of course, the situation is more problematic. In this case our epistemic access is much less direct. Toraldo di Francia makes the point quite nicely when he recalls a passage by John Herschel:

> In captain Head's amusing and vivid description of his journey across the Pampas of South America occurs an anecdote quite in point. His guide once suddenly stopped him, and, pointing high into the air, cried out: 'A lion!'. Surprised at such an exclamation, accompanied with such an act, he turned up his eyes, and with difficulty perceived, at an immeasurable height, a flight of condors soaring in circles in a particular spot. Beneath that spot, far out of sight of himself or guide, lay the carcass of a horse, and over the carcass stood (as the guide well knew) the lion, whom the condors were eyeing with envy from their airy height. The signal of the birds was to him what the sight of the lion alone could have been to the traveler, a full assurance of its existence.[1103]

In the context of physics, the apparatus and computer images act as our 'guide', indicating the observable phenomena which reveal the presence of the particles but in this case, of course, we cannot approach and see the 'lion'. Our only access is via the theory itself.

But at least in the case of classical objects, even though we cannot 'approach' them in the manner of lions, we can at least export into the micro-realm the same metaphysics of individuality that we find applicable at the macroscopic level and construct an appropriately similar semantics for our theories. In the case of quanta, even doing that is problematic.

Returning to our consideration of the Fock space approach, when we say that a vector $v \in \mathscr{H}^n$ above stands for something related to a collection of $n$ indistinguishable bosons, what kind of association are we considering? Does such a vector associate itself 'directly' with a collection of quanta in the laboratory as a name is associated with a person? It is difficult to say yes, for we simply cannot identify the particular quanta to name or to which we can make reference. We need a theory describing them in order that our logical

[1102] This is not to deny of course that there are numerous philosophical issues concerning reference in such cases, as we have indicated in Chapter 5.

[1103] Herschel 1830/1987, p. 84; Toraldo di Francia 1986, pp. 117–18.

language can be applied. In short, we don't think that in these cases we can say that a vector or other mathematical entity described in the quantum formalism applies 'directly' to the physical world. We need a mathematical description of this world first, in structural terms say, in order to construct the semantic counterpart of our theories.

This can be summed up in the following terms: given a certain domain of knowledge $\Delta$ of the empirical sciences, we describe $\Delta$ in mathematical terms by means of a mathematical model $M$, usually constructed within set theory. Then we may ask for the underlying logic $L$ of $M$, which indirectly refers to $\Delta$. Suppose $\Delta$ concerns people. Then $L$ may contain individual constants which act as names for the people, and in practice we don't need $M$ to make the link between $L$ and $\Delta$; an individual constant $p$ may stand 'directly' for 'Peter', which should refer unambiguously to Peter. But this is due to the fact that we *know* what we are talking about, at least in principle (we can point to 'Peter', for example), and $L$ should be taken to be classical logic for all the usual purposes. When it comes to quanta (and other unobservable entities, such as strings and branes), the situation is quite distinct, as $M$ seems to be absolutely necessary, for in practice, again, in a certain sense the mathematical model *is all we have*.

So, in considering quanta, we need a mathematical description of these entities, and if our aim is to consider them as indistinguishable non-individuals, as entities devoid of numerical identity, of course we should introduce something like quasi-set theory.

On these grounds, we suggest that perhaps we can provide a semantics in terms of quasi-set theory, by reading the 'crucial' vector $|n(1)\ n(2)\ \ldots\ n(n)\rangle$ as representing a quasi-set with quasi-cardinality $n$ whose elements are equivalence classes of indistinguishable $m$-atoms with quasi-cardinalities respectively $n(i)$ ($i = 1, \ldots, n$). Speculating, perhaps we could say that what we need here is a mathematical description of quantum physics similar to that one given by McKinsey *et al.* for classical particle mechanics, where they characterized a structure by posing first a non-empty set $P$ which stands for the set of particles,[1104] in order to 'know' what we are talking about. In the quantum case, we have only vectors or similar things, that is, abstract mathematical artifacts and quasi-set theory may provide the appropriate framework for retaining the mathematical structure with an interpretation in terms of 'collections of quanta'.

[1104] See Suppes 1957; 1967, 2002.

Of course, in defending the use of the apparatus of the theory $\mathfrak{Q}$, we are not endorsing any kind of *(quasi-)set-ontology*, in the sense that physical objects should be identified with a (quasi-)set.[1105] But the point is that in considering quanta or even some kind of ersatz-ontology, for instance by supposing some kind of universals instantiated by indistinguishable particles, we still need to talk of such 'particles' when answering the question: "Instantiated by what?". Whatever the final ontology we decide upon, the basic entities must be *countable*, but not *numerable*. However, we must insist that we cannot simply *suppose* that they are so, but we must also be able to provide the grounds for indicating how this is possible. Within standard set theories, where the concept of cardinal is usually defined by means of ordinals, this is of course not possible, unless we consider an alternative definition of cardinal, perhaps in the sense of Frege. But even if we consider such a possibility within standard set theories, we would be adopting again the traditional—and in this context, inappropriate—theory of identity.[1106]

So, by taking a quasi-set instead of a standard set to begin with, we can begin to carry out Post's insight that we should consider *ab ovo* indiscernible objects 'from the word go'. A pure quasi-set of objects to which no identity criteria is ascribed from the beginning gives us a more suitable way of looking at the 'semantic' aspect of the vectors (and of the quantum formalism itself). Furthermore, we recall that quasi-sets differ from quasets in certain relevant aspects (cf. Chapter 7). For instance, the latter were taken by the proponents of the theory as supporting their view that microphysics is 'a world of intensions'.[1107] But in quasi-set theory, we have a different situation: quasi-sets of indistinguishable objects can be viewed 'extensionally', although subject to our 'weak' axiom of extensionality. This kind of subtle distinction is important: whereas with quasets we have a primitive relation of $\notin$, which expresses our ignorance as to whether a certain (well-defined, in the sense that it obeys the 'traditional' theory of identity) object belongs or not to a quaset (hence the quasets have some similarity with fuzzy sets in this sense), quasi-sets have 'well-defined' borderlines, as we have suggested. In other words, in quasi-set theory

[1105] Here we follow Simons 2002, p. 38.

[1106] Johanna Seibt recently has endorsed the idea that the characterization of quanta as non-individuals is the first step in the direction of a formal semantics for quantum field theory. Her disagreement with some of the details of our framework appears to be the result of her focusing on just one of our papers, which leaves undeveloped certain mathematical details presented elsewhere. See Seibt 2002.

[1107] Dalla Chiara and Toraldo di Francia 1993.

one has the situation that a certain *quasi-object*[1108] belongs or not to a quasi-set (roughly speaking, the counter-domain of its characteristic function—if this concept is employed—is the set { 0, 1 }, and not the real interval [0,1] as in fuzzy set theory—and, in a certain sense, in quaset theory), but due to the vague aspect of such an 'object', there is no way of asserting that *this* object is that one we are talking about, as we have already noted. In short, quasi-set theory may be used as a theory of extensional *quasi-objects*.[1109]

Returning to our theme, we can say that the usual assertion that 'the vector $|\psi\rangle$' represents a collection of $n$ indistinguishable quanta (a similar case may be made in connection with the Fock space vector treated above) is a form of discourse which needs an adequate mathematical structure in which these $n$ quanta are to be represented, in order to make the standard-like semantic rules work (of course we are speaking here of a 'quasi-set semantics'). In short, except when we are considering a kind of informal semantics, as we usually do, we need something like quasi-sets in the metamathematics.[1110]

In conclusion, let us recall that we began this chapter by reviewing the problems, both physical and metaphysical, that afflict the particle interpretation of QFT and we indicated the costs involved in maintaining it. However, even if one were to insist that the 'essential reality' is a set of fields,[1111] yet still their 'particle grin' cannot be dismissed.[1112] Granted that quanta are not well defined in situations where we have interaction, it is the number of quanta in an 'aggregate' (that is, the cardinality of the appropriate quasi-set) that is typically measured in a scattering experiment, for example. As Auyang puts it, the eigenvalues $n(k)$ of the number operator $N_i(k)$ (defined in the above formalism as usual, that is, $N_i =_{df} a_i^\dagger a_i$) are 'partial manifestations' of the properties of a free field in experiments.[1113]

Questions as to the 'reality' of such quanta may be sidestepped in favour of the claim that they are, at least, 'objective'. But objective what? Substances? Trope bundles? Again each option comes with a price attached but at least the quanta package is less expensive than might be thought. 'Are field quanta

[1108] Lowe 1998, p. 71. Although Lowe disagrees in that there would be no reason to suppose that the relation of identity should make no sense for quasi-objects, it is difficult to see how one could provide the mathematical details of such a theory without this hypothesis or without introducing *ad hoc* symmetry conditions.

[1109] So, Bitbol's remark that we should have a theory which enables us to deal with extensionally given collections of indiscernible objects finds an echo in quasi-set theory; see Bitbol 2002.

[1110] Let us recall again that Dalla Chiara and Toraldo di Francia have proposed their *quasets* for the same purposes.

[1111] Weinberg 1977.

[1112] Redhead 1988.

[1113] op. cit., p. 160.

entities?' asks Auyang; "If an entity is a this-something, then field quanta are not entities; they lack numerical identities".[1114] However, an entity does not have to be a 'this-something', or possess 'primitive thisness'; as Barcan Marcus has put it:[1115]

> … all terms may 'refer' to objects, but (…) not all objects are things, where a thing is at least that about which it is appropriate to assert the identity relation. (…) If one wishes, one could say that object-reference (in terms of quantification) is a wider notion than thing-reference, the latter being also bound up with identity and perhaps with other restrictions as well, such as spatiotemporal location.

Thus, just as the Scholastics separated distinguishability from individuality, we insist on a further conceptual distinction between entities and individuals. Indeed, it is hard to see what sense can be made of the notion of 'quanta' without such a distinction. Recalling our naïve considerations of reference, the distinction maps on to that made by Barcan Marcus, between 'object' and 'thing'-reference. As she notes, if the distinction is denied at this level then it pops up again at the level of the denotation of names. If reference is understood in this wider fashion, then it simply makes no sense to assert that '$x = y$' where '$x$' and '$y$' 'refer' to non-individual entities, and of course this result is precisely captured by quasi-set theory.[1116] In our terms, 'entity' is a wider notion than 'individual' and field quanta are precisely examples of non-individual entities which are objective in the sense of being partially manifested in experiments.

We insist: the 'particle grin' remains (indeed, an empiricist might be inclined to propose this as the 'essential reality').[1117] How then are we to understand its peculiar nature? A step towards such understanding can be achieved by situating it in an appropriate formal framework[1118] and it is the beginnings of such a framework that we hope to have delineated here. We leave the final word to Auyang:[1119]

> Other people reject quantum objects because they are different but all their argument shows is that there is nothing like classical objects in the quantum realm, not that there is no quantum object.

[1114] Redhead 1988, p. 159.

[1115] Barcan Marcus 1993, p. 25.

[1116] We are grateful to Paul Teller for pressing us on this point.

[1117] "The representation of QFT has a structure compatible with atomism, and that remains true however you decide to express the states" (Huggett op. cit., p. 74).

[1118] Bain's work can be understood as an attempt to situate the particle 'grin' in an appropriate theoretical framework (Bain op. cit.).

[1119] op. cit., p. 5.

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